MANUAL

OF

PLANE TRIGONOMETRY.

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PREFACE.

My chief object in writing the present work is to lay before the student in a plain and compact form all that is necessary in the subject towards helping him to obtain a First Class Pass (Stages II., III.) in the May Examinations conducted by the Science and Art Department, South Kensington. Having had several years' experience in teaching the subject, I found, as a rule, that students required a small work dealing only with what was absolutely necessary for their examination, as the ordinary standard Treatises on Trigonometry were, on the one hand, too expensive for them, and, on the other, dealt with matter far beyond their requirements. This volume, it is to be hoped, will clear away those difficulties. I have striven to make the matter clear and intelligible, and how far I have succeeded remains to be seen.

At the end of each chapter I have given the solutions in full of a number of problems that were given at the May Examinations during the last four or five years, besides giving hints on others; so that the student may make himself familiar with the manner and principle on which such questions are solved.

The examples have been carefully graduated, and it is to be hoped the answers will be found correct. The Author will feel obliged for any corrections made through the Publisher.

J. HENCHIE.

United Westminster Schools.
January 1st, 1877.

SECOND EDITION.

The present Edition of the Manual of Plane Trigonometry embodies the result of a thorough revision of the original work. The errata which, to the Author's regret, had escaped his eye in the former Edition have now been removed or corrected. To those gentlemen, and especially to R. Tucker, M.A., of University College School, who were good enough to point out the errors they had discovered in the old Edition, or who offered suggestions for the improvement of the book, the Author takes the present opportunity of tendering his sincere thanks, and trusts that in this new Edition his friends will not find that any error of moment has been overlooked. The exhaustion of the first impression having sufficiently demonstrated that the work supplies a real want, the Author commends to the attention of teachers and beginners in Trigonometry this Second (revised and corrected) Edition of his Manual with increased confidence in its utility.

November, 1877.

CONTENTS.

PLANE TRIGONOMETRY.

CHAPTER I.

DEFINITION—GRADES AND DEGREES.

1. By "Plane Trigonometry" is meant the science which investigates the relations existing between the sides and angles of a plane triangle. Plane Trigonometry also treats of the formulæ connecting the relations of plane angles with one another, whether they form the angles of a plane triangle or not.

2. A right angle is divided into 90 equal parts called *degrees;* each of these is again divided into 60 equal parts called *minutes,* and a minute is divided into 60 equal parts called *seconds.*

The symbols denoting each of these respectively are $^\circ$ $'$ $''$; thus $57^\circ\ 17'\ 45''$ is read 57 degrees, 17 minutes, 45 seconds.

This division of an angle is called the *sexagesimal method.*

3. According to the French method, a right angle is divided into 100 equal parts called *grades;* a grade into 100 equal parts called *minutes;* a minute into 100 equal parts called *seconds.* The symbol for each is ^{g} $^\backprime$ $^{\backprime\backprime}$; as, for example, $12^g\ 15^\backprime\ 75^{\backprime\backprime}$ means

12 grades, 15 minutes, 75 seconds.

This division of angles is called the *centesimal method.*

We have also another mode of measuring angles, and which we shall refer to shortly, called *circular measure.*

We can express the minutes and seconds in the centesimal method, as the decimal fraction of a grade, very conveniently, since a minute is the hundredth part of a grade, and a second the hundredth part of a minute. Thus the above might be written $12^g \cdot 1575$.

4. The two methods might be compared, and we can thus express degrees in terms of grades, and *vice versâ*.

Let D = number of degrees in any angle,
and G = number of grades in same angle;

then we have the proportion—

$$D : G :: 90 : 100$$

$$\therefore \quad D = \frac{9}{10} G = G - \frac{1}{10} G \ldots\ldots (\alpha)$$

Also from the same proportion—

$$G = \frac{10}{9} D = D + \frac{1}{9} D.$$

Thus we get the following rules:—

(α.) *To reduce grades to degrees, we must subtract one-tenth of the number of grades from the whole, and the remainder gives us the number of degrees.*

(β.) *To reduce degrees to grades, we must add one-ninth of the number of degrees to the whole, and the sum gives us the number of grades.*

We give an example of each:—

Express $85^g\ 5^{\backprime}\ 45^{\backprime\backprime}$ in degrees.
Now $85^g\ 5^{\backprime}\ 45^{\backprime\backprime} = 85^g \cdot 0545$.

$$\begin{array}{lr}
 & \therefore\ 85{\cdot}05450 \\
\text{Sub. } \frac{1}{10}\text{th of this} = & \underline{8{\cdot}50545} \\
\text{Number of degrees} = & 76{\cdot}54905 \\
 & \underline{60} \\
 & 32{\cdot}94300 \\
 & \underline{60} \\
 & 56{\cdot}580
\end{array}$$

Ans. $76^\circ\ 32'\ 56''{\cdot}58$

Express $76° \ 32' \ 56''\cdot58$ in grades.

$$\begin{array}{r|l} 60 & 56\cdot58 \\ \hline 60 & 32\cdot943 \\ \hline & 76\cdot54905 \end{array}$$

$$\begin{array}{rl} \text{Add } \tfrac{1}{9}\text{th of this } = & 8\cdot50545 \\ \text{Number of grades } = & \overline{85\cdot05450} = 85^{g} \ 5^{\backprime} \ 45^{\backprime\backprime} \end{array}$$

5. We might also express English minutes in terms of French minutes, and English seconds in terms of French seconds.

Let M = number of English minutes in any angle, and m = number of French minutes in same angle.

$$\therefore \ M : m :: 90 \times 60 : 100 \times 100$$

$$\therefore \quad M = \frac{90 \times 60}{100 \times 100} m = \frac{27}{50} m.$$

$$\text{Also } m = \frac{50}{27} M.$$

Again, if S = number of English seconds in any angle, and s = number of French seconds in same angle, it may be found that

$$S = \frac{81}{250} s.$$

$$\text{And} \quad s = \frac{250}{81} S.$$

Exercises.

1. Express $38° \ 6' \ 39''\cdot74$ in grades, &c.
2. Express $65° \ 15' \ 20''$, $30° \ 25' \ 35''$, $48° \ 30' \ 30''\cdot75$ in grades.
3. Find how many degrees, minutes, and seconds in each of the following:—

$$66\tfrac{2}{3}^{g}, \ 10^{g}, \ 90^{g} \ 75^{\backprime} \ 25^{\backprime\backprime}\cdot75.$$

4. How many sides has a polygon which contains as many grades in all its angles as there are degrees in the angles of a twelve-sided figure?
5. The number of sides of two regular polygons are in the ratio of $9 : 6$, and the number of degrees in an angle of the one is equal

to the number of grades in an angle of the other; find the number of sides in each polygon.

N.B.—In any polygon, if n = number of sides, then (*Euc.* I. 32 Cor.) $2n - 4$ = number of right angles in all its angles $\therefore$ $\frac{2n-4}{n} \times 90$ = number of degrees in each angle.

6. Represent $31^\circ\ 45'\ 57''$ in the centesimal scale.

7. The number of sides of two regular polygons are as $2:3$; the number of grades in an angle of one equals the number of degrees in an angle of the other. Find the number of sides in each.

8. Express each of the angles of a regular polygon of $2n$ sides in grades and degrees.

9. Divide $66\frac{2}{3}^g$ into two parts, so that the number of grades in one part shall bear to the number of degrees in the other part the ratio of $5:9$.

10. The sum of two angles is 60^g and their difference is 18°; find them.

11. If the unit of angular measurement be x°, find an expression for an angle of y^g.

12. What is meant by the centesimal and sexagesimal divisions of an angle?

13. Express the sum of the angles of a regular polygon of n sides in grades and degrees.

14. One angle of a triangle contains as many degrees as another contains grades, and the third angle is half the sum of the other two; find the number of degrees in each angle.

15. The angle which is subtended by an arc equal in length to the radius is $206264{\cdot}8''$. Reduce this to grades and decimals, and thence deduce the value of $\frac{1}{\pi}$

16. Reduce the following to degrees, &c.:—

$20^g \quad 33\frac{1}{3}^g \quad 40^g \quad 50^g \quad 70^g \quad 95^g$
$25^g\ 44'\ 89''$

17. Reduce to grades, &c.:—

(*a*) $31^\circ\ 45'\ 57''$
(*b*) $11^\circ\ 15'$
(*c*) $22^\circ\ 30'$
(*d*) 75°
(*e*) $78^\circ\ 45'$

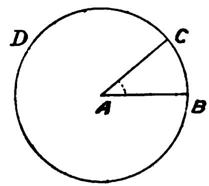
18. The number of sides of one regular polygon exceeds those of another by 1, and an angle of one exceeds an angle of the other by 4°; find the number of sides in each.

19. One regular figure has twice as many sides as another, and each of its angles greater than each of the angles of the other in the ratio of 4 : 3; find the number of sides in each.

CHAPTER II.

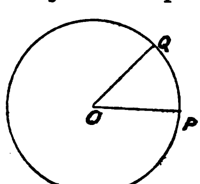
CIRCULAR MEASURE.

1. **Circular Measure.**—In the preceding chapter reference was made to this mode of estimating angles, and we now offer an explanation of the term. Suppose we take any line AB, and, with A as centre and AB as radius, describe a circle BCD. Draw any other radius AC, then the ratio $\frac{BC}{AB}$ is the circular measure of the angle BAC. Thus *the circular measure of any angle is represented by the ratio of the arc which subtends it to the radius of the circle.* If the arc be denoted by a, and the radius by r, then the circular measure of angle $= \frac{a}{r}$.



2. **Angular Unit.**—*The angular unit is that angle at the centre of a circle which is subtended by an arc equal in length to the radius of the circle.*

Take any point O, and, with this as centre and a line OP as radius, describe a circle. Cut off the arc PQ equal in length to OP, then this angle POQ is called the angular unit; and all other angles are determined by the ratio which they bear to this unit angle.



Before finding the number of degrees, &c., in this unit, let us first find a trigonometrical expression for the circumference of a circle. If C and D be the circumference and diameter respectively, then the following is a standard relation between them :—

$$C : D :: 3{\cdot}14159 : 1$$

$$\text{Or } C : 2r :: \pi : 1$$

$$\therefore C = 2\pi r$$

$$\text{Where } \pi = 3{\cdot}14159.$$

Now let A = number of degrees in angular unit, then as angles in the same circle are to one another as their subtending arcs—

$$\therefore A : 360 :: r : 2\pi r$$

$$\therefore A = \frac{360r}{2\pi r} = \frac{180}{\pi} = \frac{180}{3{\cdot}14159}$$

$$\therefore A = 57^\circ{\cdot}29577951.$$

3. From (1) of the present chapter, we can find the circular measure of any number of right angles very conveniently.

Cir. meas. of 4 right angles $= \dfrac{2\pi r}{r} = 2\pi = \dfrac{4\pi}{2}$

" " 3 " " $= \dfrac{3}{4}.\dfrac{2\pi r}{r} = \dfrac{3\pi}{2}$

" " 2 " " $= \dfrac{\pi r}{r} = \pi = \dfrac{2\pi}{2}$

" " 1 " " $= \dfrac{\pi r}{2r} = \dfrac{\pi}{2}$

$\therefore$ " " n " " $= \dfrac{n\pi}{2}$

where n is any number, integral or fractional.

Thus the cir. measure of 50 right angles $= \dfrac{50\pi}{2} = 25\pi$

4. We shall now establish a formula for expressing

any angle in degrees in terms of its circular measure, and *vice versâ.*

Let A° be any angle,
and θ its circular measure.

Now $\frac{A}{180}$ is the ratio of this angle to two right angles.

$$\text{So also is } \frac{\theta}{\pi}$$

$$\therefore \frac{A}{180} = \frac{\theta}{\pi}$$

$$\therefore A = \frac{180\ \theta}{\pi}$$

$$\text{and } \theta = \frac{A\pi}{180}$$

(α) Thus to find the circular measure of 60°.

$$\theta = \frac{60\pi}{180} = \frac{\pi}{3}$$

(β) The circular measure of an angle is $\frac{2\pi}{3}$; find it.

$$A = \frac{180}{\pi} \cdot \frac{2\pi}{3} = 120^\circ$$

This (α) may be shown to correspond with what we before had in (3).

$$\text{for since } 60^\circ = \frac{2}{3} \text{ of a right angle}$$

$$\therefore \text{ its cir. measure } = \frac{2\pi}{3} \div 2 = \frac{\pi}{3}$$

5. When any angle is given in grades, we might find its circular measure by first reducing it to degrees and then use the formula—

$$\theta = \frac{A\pi}{180}$$

Or we might proceed differently.

Let A be the angle expressed in grades, and θ its circular measure.

Now since $180° = 200^g$

$$\therefore \frac{A}{200} = \frac{\theta}{\pi}$$

$$\therefore \theta = \frac{A\pi}{200}$$

$$\text{and } A = \frac{200}{\pi}\theta$$

These results should be very carefully committed to memory by the student, so that he may be able to put them into practice at any moment when necessary.

The following exercises may be worked from the information given in this chapter.

Exercises.

1. What is the angular unit? Find the number of grades, minutes, and seconds contained in it.

2. Show how to find the circular measure of n right angle, where n is any number. Work out the formulæ connecting any angle expressed in degrees with the same angle expressed in circular measure.

3. Give a trigonometrical expression for the circumference of a circle. What number is denoted by π?

4. Find the circular measure of the following angles:—

$$30° \quad \tfrac{4}{15} \text{ of a right angle} \quad 500^g$$
$$1° \quad 10'' \qquad\qquad \pi^g$$

5. Find the angles of which the following are the respective circular measures: $\frac{\pi}{4}$, $\frac{3\pi}{2}$, $\frac{11}{21}$, $\pi+1$, $\frac{3}{4}$.

6. One angle of a triangle is π grades, and another is π degrees; find the circular measure of the third.

7. If the number of degrees in an angle be equal to the number of grades in its complement, find the circular measure of this angle.

8. If an angle of $57°{\cdot}2958$ be represented by the number $5{\cdot}09296$, what is the unit of angular measurement?

9. If the circumference of a circle be divided into x equal parts, how many of those would the arc opposite the angular unit contain?

10. Find the number of degrees in an angle at the centre of a circle, of diameter 24 feet, subtended by an arc of 8 inches.

11. Show that $G : 100 :: 2\theta : \pi$,

where G = number of grades in an angle,

and θ = its circular measure.

12. If the measure of an angle be 2 when the angular unit is $\frac{2}{3}$ of a right angle, find the ordinary circular measure of the same angle.

13. Find an equation connecting A, a, r, when the angle which subtends an arc equal in length to the radius is $\frac{1}{n}$th of the angular unit; where A, a, r, stand for angle, arc, and radius respectively.

14. Find the circular measure of $17° 27' 12''$.

15. At what distance from an observer's eye will a coin $\frac{7}{8}$ths of an inch in diameter completely cover the moon's disc considered to subtend an angle of $1°$.

16. If N = number of seconds in any angle, then

$$N = 206265'' \times \text{circular measure.}$$

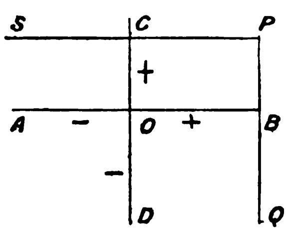
17. Find the circular measure of $18°$.

18. The angle which the diameter of the earth subtends at the sun's centre is $17''\frac{1}{5}$. Find the distance of the sun from the earth, the diameter of earth = 7926 miles.

CHAPTER III.

ON SIGNS + & −, AND TRIGONOMETRICAL RATIOS.

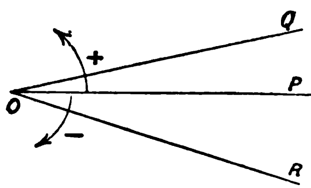
1. The following method is found useful in reference to *positive* and *negative* lines. Let two lines AB, CD cut one another at right angles in the point O; then if this point be considered as the point of origin, all lines measured *upwards* and to the *right* of O are considered *positive* or +; and all lines measured *downwards* and to the *left* of O are considered *negative* or − lines. If any line be



parallel with AB and CD, as in the figure, then CP, PB are *positive* also, BQ and SC *negative*.

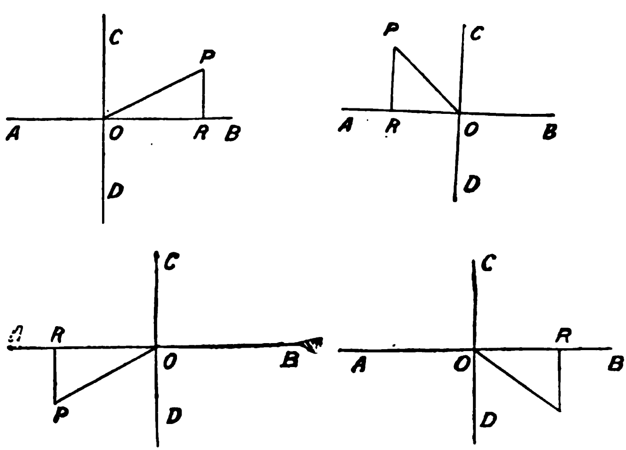
2. A similar convention is made as regards angles.

Let a line OQ revolve from the position OP, *upwards* and around the point O, or in the opposite direction to the hands of a clock. These angles are considered as *positive*.



If the line revolve in the downward direction, the angles it traces out are considered as *negative*. If the angles POQ and POR be equal to one another, and each be denoted by A, then

$$\angle\ POQ = A \text{ and } \angle\ POR = -A.$$



3. In each of the above figures let the revolving line OP come into the four positions shown. Let fall PR perpendicular to AB.

Then $\frac{PR}{OP}$ is the sine of the $\angle$ BOP

$\frac{OR}{OP}$ " cosine "

$\frac{PR}{OR}$ " tangent "

$\frac{OR}{PR}$ " cotangent "

$\frac{OP}{OR}$ " secant "

$\frac{OP}{PR}$ " cosecant "

If we denote this angle by A, then

$$\sin A = \frac{PR}{OP} \qquad \cot A = \frac{OR}{PR}$$

$$\cos A = \frac{OR}{OP} \qquad \sec A = \frac{OP}{OR}$$

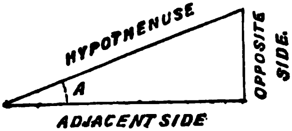
$$\tan A = \frac{PR}{OR} \qquad \operatorname{cosec} A = \frac{OP}{PR}$$

$$\operatorname{vers} A = 1 - \cos A$$

$$\operatorname{covers} A = 1 - \sin A$$

4. So that if we have a right-angled triangle, it will be well for the student to commit to memory the following:—

Let A be the angle, then



$$\sin A = \frac{\text{opposite side}}{\text{hypothenuse}}$$

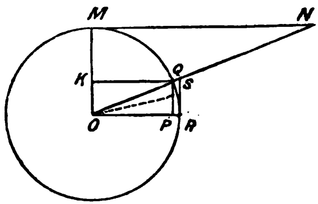
$$\cos A = \frac{\text{adjacent side}}{\text{hypothenuse}}$$

$$\tan A = \frac{\text{opposite side}}{\text{adjacent side}} \qquad \sec A = \frac{\text{hypothenuse}}{\text{adjacent side}}$$

$$\cot A = \frac{\text{adjacent side}}{\text{opposite side}} \qquad \operatorname{cosec} A = \frac{\text{hypothenuse}}{\text{opposite side}}$$

5. From the following figure we might show what is meant by sin, cos, tan, &c.

Let QR be any arc of a circle and complete the figure as here shown.



Then QP is called the sine of the arc QR

OP	is called its	cosine
RS	„	tangent
MN	„	cotangent
OS	„	secant
ON	„	cosecant
PR	„	versed sine
MK	„	coversed sine

Now, considering the angle POQ as A

$$\text{then } \sin A = \frac{\sin \text{ arc}}{\text{radius}}$$

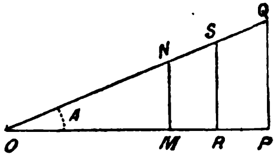
$$\cos A = \frac{\cos \text{ arc}}{\text{radius}}$$

and so on for each of the other ratios.

6. If we joined QR and let fall a perpendicular OT on it, it will of course bisect QR in the point T.

now QR is the chord of the arc
and QT is the sin of half the arc
$\therefore$ chord arc $= 2$ sin half arc
and chord arc $= 2 \sin \frac{1}{2} A \times r$.

7. The sin, cos, tan, &c., of an angle remain the same so long as this angle is not altered.



$$\text{For } \sin A = \frac{MN}{ON}$$

$$\text{and } \sin A = \frac{RS}{OS}$$

$$\text{but } \frac{MN}{ON} = \frac{RS}{OS}$$

by similar triangles, or sin A remains unaltered. Similarly with the other functions.

8. In any right-angled triangle OPQ (figure to the last article)

since $\sin A = \dfrac{PQ}{OQ}$ and $\operatorname{cosec} A = \dfrac{OQ}{PQ}$

by multiplication, $\sin A \cdot \operatorname{cosec} A = 1$

$$\therefore \sin A = \frac{1}{\operatorname{cosec} A}$$

$$\text{and } \operatorname{cosec} A = \frac{1}{\sin A}$$

$$\text{similarly } \sec A = \frac{1}{\cos A}$$

$$\cos A = \frac{1}{\sec A}$$

$$\tan A = \frac{1}{\cot A}$$

$$\cot A = \frac{1}{\tan A}$$

$$\text{Again, since } \tan A = \frac{PQ}{OP} = \frac{\dfrac{PQ}{OQ}}{\dfrac{OP}{OQ}} = \frac{\sin A}{\cos A}$$

$$\text{and also } \cot A = \frac{\cos A}{\sin A}$$

$$\text{Again, } OQ^2 = PQ^2 + OP^2 \quad (\text{I.47}) \quad . \quad . \quad . \quad (\alpha)$$

$$\therefore 1 = \left(\frac{PQ}{OQ}\right)^2 + \left(\frac{OP}{OQ}\right)^2$$

$$1 = \sin^2 A + \cos^2 A \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

By dividing each side of this equation (α) by OP^2

$$\text{we get } \left(\frac{OQ}{OP}\right)^2 = \left(\frac{PQ}{OP}\right)^2 + 1$$

$$\therefore \sec^2 A = \tan^2 A + 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

similarly by dividing by PQ^2 we get

$$\operatorname{cosec}^2 A = 1 + \cot^2 A \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

We could, in fact, get these equations at once by inspecting the figure to Art. 5.

From these we get the following:—

$$\sin^2 A = 1 - \cos^2 A \therefore \sin A = \sqrt{1 - \cos^2 A}$$
$$\cos^2 A = 1 - \sin^2 A \therefore \cos A = \sqrt{1 - \sin^2 A}$$
$$\sec^2 A = 1 + \tan^2 A \therefore \sec A = \sqrt{1 + \tan^2 A}$$
$$\tan^2 A = \sec^2 A - 1 \therefore \tan A = \sqrt{\sec^2 A - 1}$$
$$\cot^2 A = \operatorname{cosec}^2 A - 1 \therefore \cot A = \sqrt{\operatorname{cosec}^2 A - 1}$$

9. By means of these expressions we can express any function in terms of any other. Let us express all the rest in terms of the *cosine*.

$$\sin A = \sqrt{1 - \cos^2 A}$$
$$\tan A = \frac{\sin A}{\cos A} = \frac{\sqrt{1 - \cos^2 A}}{\cos A}$$
$$\cot A = \frac{\cos A}{\sin A} = \frac{\cos A}{\sqrt{1 - \cos^2 A}}$$
$$\sec A = \frac{1}{\cos A}$$
$$\operatorname{cosec} A = \frac{1}{\sin A} = \frac{1}{\sqrt{1 - \cos^2 A}}$$
$$\operatorname{vers} A = 1 - \cos A$$
$$\operatorname{covers} A = 1 - \sin A = 1 - \sqrt{1 - \cos^2 A}.$$

10. All the preceding results in Articles 8 and 9 of this chapter should be carefully committed to memory. The student should work out and remember the following expressions also, as they are constantly called for in practice:—

$$\sin A = \frac{\tan A}{\sqrt{1 + \tan^2 A}} \qquad \cos A = \frac{1}{\sqrt{1 + \tan^2 A}}$$
$$\sin A = \frac{\sqrt{\sec^2 A - 1}}{\sec A}$$
$$\tan A = \frac{\sin A}{\sqrt{1 - \sin^2 A}} \qquad \sec A = \frac{1}{\sqrt{1 - \sin^2 A}}$$

EXERCISES.

1. If $\tan A = \frac{2}{3}$, find the other functions.

$$\text{Now } \sin A = \frac{\tan A}{\sqrt{1 + \tan^2 A}} = \frac{\frac{2}{3}}{\sqrt{1 + \frac{4}{9}}} = \frac{\frac{2}{3}}{\frac{\sqrt{13}}{3}} = \frac{2}{\sqrt{13}}$$

$$\cos A = \frac{1}{\sqrt{1+\tan^2 A}} = \frac{1}{\sqrt{1 + \frac{4}{9}}} = \frac{3}{\sqrt{13}}$$

$$\cot A = \frac{1}{\tan A} = \frac{3}{2}$$

$$\sec A = \frac{1}{\cos A} = \frac{\sqrt{13}}{3}$$

$$\operatorname{cosec} A = \frac{1}{\sin A} = \frac{\sqrt{13}}{2}$$

Similar questions to this can always be worked by the same method.

2. If $\sec A = \sqrt{\frac{3}{2}}$, find the other functions of A.

3. The cosecant of an angle is $\frac{5}{3}$; find the other functions of the angle.

4. If $\cos A = \frac{5}{13}$, find the other functions of A.

5. If $\tan A = \frac{1}{x}$, find $\sin A$.

6. If $\frac{\sin A}{\cos A} = \frac{a}{b}$, find $\sin A$ in terms of a and b.

$$\text{Now } \frac{\sin A}{\sqrt{1 - \sin^2 A}} = \frac{a}{b}$$

$$\therefore \frac{\sin^2 A}{1 - \sin^2 A} = \frac{a^2}{b^2}$$

$$\therefore a^2 - a^2\sin^2 A = b^2\sin^2 A$$

$$\therefore \sin^2 A(a^2 + b^2) = a^2$$

$$\therefore \sin A = \pm \frac{a}{\sqrt{a^2 + b^2}}$$

7. Prove that

$$\tan A + \cot A = \sec A \, . \operatorname{cosec} A$$

$$\text{Now } \tan A + \cot A = \frac{\sin A}{\cos A} + \frac{\cos A}{\sin A}$$

$$= \frac{\sin^2 A + \cos^2 A}{\sin A \,.\, \cos A}$$

$$= \frac{1}{\sin A \,.\, \cos A} = \sec A \,.\, \operatorname{cosec} A.$$

8. Prove that $\sec^2 A + \operatorname{cosec}^2 A = \sec^2 A \,.\, \operatorname{cosec}^2 A$

$$\text{Now } \sec^2 A + \operatorname{cosec}^2 A = \frac{1}{\cos^2 A} + \frac{1}{\sin^2 A}$$

$$= \frac{\sin^2 A + \cos^2 A}{\sin^2 A \,.\, \cos^2 A} = \frac{1}{\sin^2 A \,.\, \cos^2 A} = \operatorname{cosec}^2 A \,.\, \sec^2 A.$$

9. Define the trigonometrical ratios of any angle. And show by one figure what is meant by sine, cosine. tangent, &c.

10. Show geometrically that $\sin 2A$ is less than $2 \sin A$.

11. If $\tan A = \cdot 97$, find $\sin A$ to two places of decimals.

12. If $\tan A = \frac{1}{2}$, find the other functions.

12. How are positive and negative angles measured? how also positive and negative lines? Illustrate by figure.

13. Show by figure that $\text{vers } A = 1 - \cos A$.

14. Prove that $\sin \frac{A}{2} = \frac{\text{chord arc}}{2 \text{ radius.}}$

15. Show that

$$\tan A = \frac{\sqrt{2 \text{ vers } A - \text{vers}^2 A}}{1 - \text{vers } A}$$

16. Given

	$\text{vers } A = \frac{1}{2}$	find $\sin A$
	$\cot A = 5$	find $\sec A$
	$\tan A = \frac{3}{5}$	find $\cos A$
	$\sec A = 3{\cdot}15$	find $\cos A$

17. Show that $\tan A = \dfrac{\sin A}{\cos A}$

18. If

$$p = a \sin A \cos B$$
$$q = a \sin A \sin B$$
$$r = a \cos A$$

Prove that

$$p^2 + q^2 + r^2 = a^2$$

19. Given $\cos^2 A - 3 \sin A = \frac{3}{4}$ find $\sin A$.

20. Find $\tan A$ from the equation,

$$x \sin A = y \cos A - y$$

21. Prove that

$$\sin^2 A \,.\, \sin^2 B + \cos^2 A \cos^2 B + \sin^2 A \cos^2 B + \sin^2 B \cos^2 A = 1.$$

22. Show that

$$\sec A - \tan A = \sqrt{\frac{1 - \sin A}{1 + \sin A}}$$

23. Prove the following:—

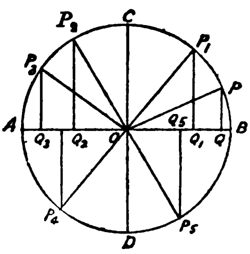
$$\sin^2 \alpha - \sin^2 \beta = \sin^2 \alpha \cos^2 \beta - \cos^2 \alpha \sin^2 \beta.$$

$$\tan^2 \alpha = \frac{\sec \alpha}{\operatorname{cosec} \alpha \cot \alpha}$$

24. Prove that so long as the angle remains the same, the trigonometrical ratios will also be the same.

CHAPTER IV.

TRACING THE ANGLE THROUGH EACH QUADRANT; ANGLES OF 30°, 45°, &c.



1. Let OP, OP_2, OP_4, OP_5, be the positions of the revolving line OP in each quadrant. Let fall perpendiculars PQ_1, P_2Q_2, &c.

Let us denote the angle BOP_1, BOP_2, BOP_4, in each quadrant by A.

Now in the 1st quadrant $\sin A = \dfrac{PQ}{PO}$

but PQ is a positive line (1. Chap. III.).

$\therefore$ in 1st quadrant sin A is positive.

In the 2nd quadrant $\sin A = \dfrac{P_2Q_2}{P_2O}$

$\therefore$ in 2nd quadrant also sin A is positive.

In the 3rd quadrant $\sin A = \dfrac{P_4Q_4}{P_4O}$

but P_4Q_4 is a negative line.

$\therefore$ in the 3rd quadrant sin A is negative.

In the 4th quadrant $\sin A = \dfrac{P_5Q_5}{P_5O}$

$\therefore$ " $\sin A$ is negative.

Now in 1st quadrant, as the angle A increases the perpendicular PQ increases also, and since the radius is a constant quantity, therefore the sine *increases*.

In the 2nd quadrant, as the angle increases the perpendicular P_2Q_2 decreases, consequently the sine *decreases*.

In a similar manner it can be seen that in the 3rd quadrant the sine *increases* and in the 4th decreases.

Again, if the revolving line OP coincided with OB, the angle A would vanish and likewise the perpendicular PQ; hence $\sin 0^\circ = 0$.

If the revolving line as it sweeps round fell on OC, the perpendicular PQ would also coincide with OC, hence in this case

$$\sin 90^\circ = \frac{OC}{OC} = 1$$

Next suppose the revolving line coincided with OA, the perpendicular P_2Q_2 would vanish, hence $\sin 180^\circ = 0$.

Again, if it coincided with OD, we would have

$$\sin 270^\circ = -\frac{OD}{OD} = -1$$

We can also see that

$$\sin 360^\circ = 0.$$

2. We can treat the *cosine* of the angle by a similar method, also the secant, and cosecant, tan, &c. We will perform the operation on the tangent, and the student will find it a useful exercise to go through the others by himself, and arrive at the results tabulated below.

In 1st quadrant $\tan A = \dfrac{PQ}{OQ}$, but

PQ and QO are each *positive;*

$\therefore$ tan A is positive.

In 2nd quadrant $\tan A = \frac{P_2Q_2}{Q_2O}$,
but P_2Q_2 is *positive* and Q_2O *negative*;
$\therefore$ tan A is negative.

In 3rd quadrant $\tan A = \frac{P_4Q_4}{Q_4O}$,
but each of those lines is *negative*;
$\therefore$ tan A is positive.

In 4th quadrant $\tan A = \frac{P_5Q_5}{Q_5O}$,
but P_5Q_5 is *negative*, while Q_5O is *positive*;
$\therefore$ tan A is *negative*.

Now in 1st quadrant, as A increases the perpendicular increases and base decreases;

hence tan A *increases*.

In 2nd quadrant, as A increases the perpendicular decreases and base increases.

$\therefore$ tan A *decreases*.

In 3rd quadrant, as A increases the perpendicular increases and the base decreases.

$\therefore$ tan A *increases*.

In 4th quadrant, as A increases the perpendicular decreases, while the base increases.

$\therefore$ tan A *decreases*.

Again, if the revolving line OP coincided with OB, the angle and perpendicular would vanish;

hence $\tan 0° = 0$

If OP coincided with OC, the perpendicular coincides with the radius, while the base vanishes;

$$\text{hence } \tan 90° = \frac{OC}{0} = \infty$$

If OP swept round further still and coincided with OA, it would trace out an angle of 180°, and the perpendicular vanishes, while the base coincides with OA; hence

$$\tan 180° = \frac{0}{-OC} = 0$$

It can also be seen that

$$\tan 270^\circ = \frac{\text{OD}}{0} = \infty$$

$$\text{and } \tan 360^\circ = \frac{0}{\text{OB}} = 0$$

3. It must be remembered that cos A is never greater than unity, and that cos A is unity when A = 0°,

since vers A = 1 − cos A

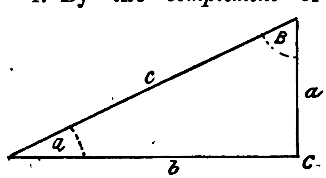
∴ vers A is always positive, and its greatest value is when A = 180°, for then cos A = − 1

∴ vers A = 2.

The following table shows at a glance the various changes in sign and magnitude of the several functions of the angle :—

	1st Quad.	Sign.	2nd Quad.	Sign.	3rd Quad.	Sign.	4th Quad.	Sign.
sin A varies from	0 to 1	+	1 to 0	+	0 to −1	−	−1 to 0	−
cos A ,,	1 ,, 0	+	0 ,, −1	−	−1 ,, 0	−	0 ,, 1	+
tan A ,,	0 ,, ∞	+	∞ ,, 0	−	0 ,, ∞	+	∞ ,, 0	−
cot A ,,	∞ ,, 0	+	0 ,, ∞	−	∞ ,, 0	+	0 ,, ∞	−
sec A ,,	1 ,, ∞	+	∞ ,, −1	−	−1 ,, ∞	−	∞ ,, 1	+
cosec A ,,	∞ ,, 1	+	1 ,, ∞	+	∞ ,, −1	−	−1 ,, ∞	−

4. By the *complement* of an angle is meant *the defect of that angle from a right angle.* Thus if A be the angle, then its complement is (90° − A).



In any right-angled triangle where C is the right angle and a and β are the other angles, then these angles are complementary, since both together make a right angle. Or if we denote one angle by a, then the other will be denoted by $(90 - a)$.

5. In the last figure

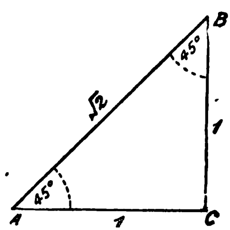
$$\sin \alpha = \frac{a}{c}, \text{ but } \cos \beta = \frac{a}{c};$$
$$\text{hence } \sin \alpha = \cos \beta,$$
$$\text{or } \sin \alpha = \cos (90^\circ - \alpha).$$

We might express this in words by saying that the "*sine of an angle is equal to the cosine of its complement.*"

It may be similarly proved that the tangent of an angle is equal to the cotangent of its complement, and secant of an angle is equal to the cosecant of its complement.

6. To find the sin, cos, &c., of an angle of 45°.

Let ACB be an isosceles right-angled triangle, having C the right angle; then it is evident that each of the angles at A and B are 45°. Suppose we agree to denote each side of this triangle by unity or 1, then the hypothenuse AB must be denoted by $\sqrt{2}$ (I. 47).

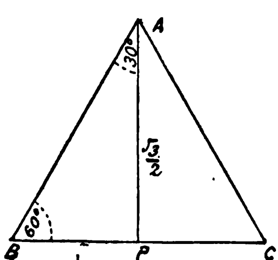


$$\text{Hence } \sin 45^\circ = \cos 45^\circ = \frac{1}{\sqrt{2}}$$
$$\tan 45^\circ = \cot 45^\circ = 1$$
$$\sec 45^\circ = \text{cosec } 45^\circ = \sqrt{2}$$

7. To find the sin, cos, &c., of an angle of 60°.

Let ABC be an equilateral triangle, then manifestly each angle is 60°.

Let fall AP perpendicular on BC, then (I. 26) BC is bisected in P, and also the angle BAC is bisected by this line AP.



Suppose we represent each side of this triangle by (1); then

$$AB = 1$$
$$BP = \tfrac{1}{2}$$
$$AP = \sqrt{1 - \tfrac{1}{4}} = \frac{\sqrt{3}}{2} \qquad \text{(I. 47)}$$

Hence

$$\cos 60^\circ = \frac{\tfrac{1}{2}}{1} = \tfrac{1}{2} \qquad \cot 60^\circ = \frac{\tfrac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}$$

$$\sin 60^\circ = \frac{\frac{\sqrt{3}}{2}}{1} = \frac{\sqrt{3}}{2} \qquad \sec 60^\circ = \frac{1}{\tfrac{1}{2}} = 2$$

$$\tan 60^\circ = \frac{\frac{\sqrt{3}}{2}}{\tfrac{1}{2}} = \sqrt{3} \qquad \operatorname{cosec} 60^\circ = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}$$

Since the angle BAP $\doteq 30^\circ$, we might find from the same figure all the trigonometrical functions of this angle.

Then,

$$\sin 30^\circ = \frac{BP}{BH} = \frac{\tfrac{1}{2}}{1} = \tfrac{1}{2}$$

$$\cos 30^\circ = \frac{AP}{AB} = \frac{\frac{\sqrt{3}}{2}}{1} = \frac{\sqrt{3}}{2}$$

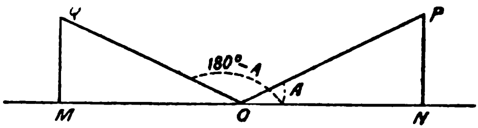
$$\cot 30^\circ = \frac{AP}{BP} = \frac{\frac{\sqrt{3}}{2}}{\tfrac{1}{2}} = \sqrt{3}$$

and similarly for the others.

Or we might otherwise find the functions of an angle of 30° from those already found of 60°, since 30° is the complement of 60°.

$$\sin 30^\circ = \cos 60^\circ = \tfrac{1}{2}$$
$$\cos 30^\circ = \sin 60^\circ = \frac{\sqrt{3}}{2}$$
$$\tan 30^\circ = \cot 60^\circ = \frac{1}{\sqrt{3}}$$
$$\&c. = \&c. = \&c.$$

8. By the *supplement* of an angle is meant its defect from two right angles; thus if A be any angle, then its supplement is $180° - A$. We will now show the relations existing between the trigonometrical functions of an angle and those of its supplement.



Let NOP be any angle A. Draw OQ, making the angle MOQ equal to it. Make $OP = OQ$, and let fall perpendiculars QM, PN, on MN.

Since the angles QON, QOM, are together equal to two right angles.

$$\text{Hence } \angle\, QON = 180° - \angle\, MOQ$$
$$\text{but } \angle\, MOQ = \angle\, A$$
$$\therefore \angle\, QON = 180° - A$$

Now the two triangles are geometrically equal in all respects (I. 26).

$$\text{Hence } PN = QM$$
$$\text{and } OM = ON$$
$$\therefore \frac{PN}{OP} = \frac{QM}{OQ}$$
$$\text{but } \frac{PN}{OP} = \sin A$$
$$\text{and } \frac{QM}{OQ} = \sin QON = \sin 180° - A$$
$$\therefore \sin A = \sin (180° - A)$$
$$\text{Again } \frac{ON}{OP} = -\frac{OM^*}{OQ}$$
$$\text{or } \cos A = -\cos (180° - A)$$

* This line OM must be considered negative, since it is measured towards the left of the point O. Having found the sin and cos, it is unnecessary for us to refer to

the figure for the others, since they can be found from those two in the following way :—

$$\tan(180^\circ - A) = \frac{\sin(180 - A)}{\cos(180 - A)} = \frac{\sin A}{-\cos A} = -\tan A$$

$$\cot(180^\circ - A) = \frac{1}{\tan(180^\circ - A)} = \frac{1}{-\tan A} = -\cot A$$

$$\sec(180 - A) = \frac{1}{\cos(180 - A)} = \frac{1}{-\cos A} = -\sec A$$

$$\operatorname{cosec}(180^\circ - A) = \frac{1}{\sin(180^\circ - A)} = \frac{1}{\sin A} = \operatorname{cosec} A$$

$$\operatorname{vers}(180^\circ - A) = 1 - \cos(180^\circ - A) = 1 + \cos A$$
$$\operatorname{covers}(180^\circ - A) = 1 - \sin(180^\circ - A) = 1 - \sin A$$

Thus the sine of any angle = sine of its supplement.

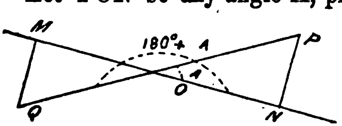
,,	cosine	,,	=	− cosine	,,
,,	tangent	,,	=	− tangent	,,
,,	cotangent	,,	=	− cotangent	,,
,,	secant	,,	=	− secant	,,
,,	cosecant	,,	=	cosecant	,,

These can easily be remembered, as the same function is in both the angle and its supplement; bearing in mind that *two* only have the + sign—namely, *sin* and *cosec.*

9. To show that $\sin A = -\sin(180^\circ + A)$

$$\cos A = -\cos(180^\circ + A)$$

Let PON be any angle A, produce PO to Q, making OQ = OP. Let fall perpendiculars QM, PN, on MN. Then the whole *positive* angle NOQ = $180^\circ + A$.



The triangles are geometrically equal (I. 26).

Hence PN = QM

ON = OM

$$\therefore \frac{PN}{PO} = -\frac{QM}{QO} \quad \text{(QM is negative, being measured downwards.)}$$

or $\sin A = -\sin(180 + A)$.

$$\text{Again } \frac{ON}{OP} = -\frac{OM}{OQ}$$

$$\text{or } \cos A = -\cos(180° + A)$$

$$\tan(180° + A) = \frac{\sin(180° + A)}{\cos(180° + A)} = \frac{-\sin A}{-\cos A} = \tan A$$

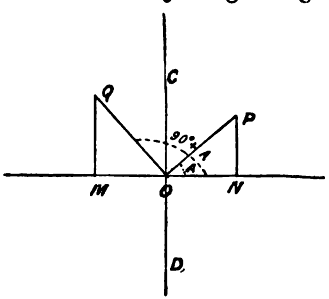
$$\cot(180° + A) = \frac{1}{\tan(180 + A)} = \frac{1}{\tan A} = \cot A$$

$$\sec(180° + A) = \frac{1}{\cos(180° + A)} = \frac{1}{-\cos A} = -\sec A$$

$$\operatorname{cosec}(180° + A) = \frac{1}{\sin(180° + A)} = \frac{1}{-\sin A} = -\operatorname{cosec} A.$$

10. To show that $\sin(90 + A) = \cos A$
and $\cos(90 + A) = -\sin A$.

Let PON be any angle A. Draw OQ at right angles to it, and make OP = OQ. Let fall the perpendiculars PN, QM. Let CD be a line through O at right angles to MN.



Now $\angle$ CON = $\angle$ QOP, each being a right angle; take away the common $\angle$ COP.

$\therefore$ $\angle$ QOC = $\angle$ PON
but $\angle$ QOC = $\angle$ MQO being alternate angles;

$$\therefore \angle MQO = \angle PON = \angle A.$$

It can be seen that the triangles are geometrically equal in all respects (I. 26).

$$\text{Hence } PN = MO$$
$$QM = ON$$
$$\therefore \frac{QM}{QO} = \frac{ON}{OP}$$
$$\text{or } \sin(90° + A) = \cos A$$

$$\text{Again} -\frac{MO}{OQ} = \frac{PN}{PO}$$

$$\text{or } -\cos(90^\circ + A) = \sin A$$

$$\therefore \cos(90^\circ + A) = -\sin A$$

$$\text{Hence } \tan(90^\circ + A) = \frac{\sin(90^\circ + A)}{\cos(90^\circ + A)} = \frac{\cos A}{-\sin A}$$

$$= -\cot A$$

$$\cot(90^\circ + A) = \frac{1}{\tan(90^\circ + A)} = \frac{1}{-\cot A} = -\tan A$$

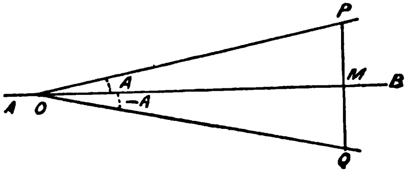
$$\sec(90^\circ + A) = \frac{1}{\cos(90^\circ + A)} = \frac{1}{-\sin A} = -\operatorname{cosec} A$$

$$\operatorname{cosec}(90^\circ + A) = \frac{1}{\sin(90^\circ + A)} = \frac{1}{\cos A} = \sec A.$$

11. To prove that $\sin(-A) = -\sin A$

$\cos(-A) = \cos A.$

Let MOP be any angle A, and MOQ be an angle drawn equal to it in the *negative* direction.



Make OP = OQ, join PQ, cutting AB in M.

The triangles are geometrically equal (I. 4).

Hence PM = MQ

$$\therefore \frac{PM}{OP} = -\frac{MQ}{OQ}$$

$$\text{or } \sin A = -\sin(-A)$$

$$\therefore \sin(-A) = -\sin A$$

$$\text{Again, } \frac{OM}{OQ} = \frac{OM}{OP}$$

$$\text{or } \cos(-A) = \cos A$$

$$\text{Hence } \tan(-A) = -\tan A$$
$$\cot(-A) = -\cot A$$
$$\sec(-A) = \sec A$$
$$\operatorname{cosec}(-A) = -\operatorname{cosec} A.$$

From the last figure it can be seen that $\frac{MQ}{QO}$ is either the sine of the *positive* angle BOQ, or the sine of the *negative* angle A.

$$\text{Hence } \sin BOQ = \sin(-A) = -\sin A$$
$$\text{or } \sin(360^\circ - A) = -\sin A$$
$$\text{and } \cos(360^\circ - A) = \cos A$$
$$\&c. = \&c.$$

Again, suppose the revolving line OP to come into the same position after one revolution, it would trace out an angle ($360^\circ + A$), and the sine of the angle would in no way be affected. It will, in fact, remain the same after any number of revolutions of OP, provided this line always comes into the same position.

$$\text{Hence } \sin A = \sin(360^\circ + A)$$
$$\text{and also } \cos A = \cos(360^\circ + A)$$
$$\&c. = \&c.$$

$$\begin{aligned}
\text{Now } \sin A &= \sin(360^\circ + A) \\
&= \sin(2 \cdot 180^\circ + A) \\
&= \sin(2n \,.\, 180^\circ + A) \text{ where } n \text{ is any integer.} \quad \ldots\ldots \quad (1)
\end{aligned}$$

$$\begin{aligned}
\text{And } \sin A &= \sin(180^\circ - A) \\
&= \sin\{2n \,.\, 180^\circ + (180^\circ - A)\} \text{ by last equation.} \\
&= \sin\{(2n+1) \,.\, 180^\circ - A\} \quad \ldots \quad (2)
\end{aligned}$$

Again, since

$$\begin{aligned}
\sin A &= -\sin(180 + A) \\
&= -\sin\{(2n+1)\, 180^\circ - (180^\circ + A)\} \\
&= -\sin(2n \,.\, 180^\circ - A) \quad \ldots\ldots \quad (3)
\end{aligned}$$

$$\begin{aligned}
\text{And } \sin A &= -\sin(-A) \\
&= -\sin\{(2n+1)\, 180^\circ - (-A)\} \\
&= -\sin\{(2n+1)\, 180^\circ + A\} \quad \ldots \quad (4)
\end{aligned}$$

12. Similarly we may find,

$$\cos A = \cos (2n \,.\, 180° + A)$$
$$\text{or } \cos A = - \cos \{(2n + 1)\, 180° - A\}$$
$$,, \ \cos A = \cos (2n \,.\, 180° - A)$$
$$,, \ \cos A = - \cos \{(2n + 1)\, 180° + A)$$

Tan A, sec A, &c., may also be found by following a like method.

The student should make it a point to commit the above formulæ to memory. This will be found no difficult task if the *starting-point* of each be borne in mind.

Thus, suppose at any time it be required to prove the following formula :—

$$\sin A = \sin \{(2n + 1)\, 180° - A\}$$

Knowing that $\sin A = \sin (2n \,.\, 180 + A) \quad . \quad . \quad . \ (\alpha)$
and that $\sin A = \sin (180° - A) \quad . \quad . \quad . \quad . \ (\beta)$

$\therefore$ in equation (α), we can for the angle A substitute $(180° - A)$ and we get

$$\sin A = \sin \{2n \,.\, 180° + (180° - A)\}$$
$$= \sin \{(2n + 1)\, 180° - A\}$$

13. All the formulæ in the latter part of 11 and 12 can be found from the figure in the same way as the preceding ones in 8, 9, 10, and 11. We will work one of them by way of illustration. If this be well understood, there will be no difficulty in treating the others by a similar method.

Remembering that all angles are measured from the initial line, and referring to the first figure of this chapter, we see that the *sines* of all angles formed by the initial line OB and the revolving line OP have the same sign on either side of the line AB. Also the *cosines* of all angles formed by the initial line OB and the revolving line OP have the same sign when this line OP is on either side of CD.

The *tangents* will also have the same sign for all

angles where the line OP is in the same or opposite quadrants.

Now the angles A and ($2n \cdot 180° + A$) always have the same *sine*, *cosine*, *tangent*, &c., since the revolving line OP comes into its own position again after any number of revolutions. Suppose now that we had to show that

$$\sin \{(2n + 1)\, 180° + A\} = -\sin A$$

Looking at the figure, we can see that when OP_1 traces out this angle $\{(2n + 1)\, 180° + A\}$, its position will fall in the 3rd quadrant as OP_4 in a direct line with its position OP_1 in the 1st quadrant.

$$\text{And that } \sin \{(2n + 1)\, 180° + A\} = \frac{P_4Q_4}{P_4O}$$

$$\text{,, } \quad \sin A = \frac{P_1Q_1}{P_1O}$$

$$\text{but } \frac{P_1Q_1}{P_1O} = -\frac{P_4Q_4}{P_4O}$$

$\therefore \sin \{(2n + 1)\, 180° + A\} = -\sin A$. Similarly

$$\cos \{(2n + 1)\, 180° + A\} = -\cos A$$

and so on for all the rest.

14. Since the circular measure of 4 right angles is 2π, and that of two right angles π, then if θ be the circular measure of A°, we might tabulate all the preceding formulæ as follows:—

$$\left.\begin{aligned} \sin \theta &= \cos \left(\frac{\pi}{2} - \theta\right) \\ \cos \theta &= \sin \left(\frac{\pi}{2} - \theta\right) \\ \tan \theta &= \cot \left(\frac{\pi}{2} - \theta\right) \\ \&c. &= \&c. \end{aligned}\right\} \text{ from Art. 5 of this Chapter.}$$

$$\left.\begin{aligned} \sin \theta &= \sin (\pi - \theta) \\ \cos \theta &= -\cos (\pi - \theta) \\ \tan \theta &= -\tan (\pi - \theta) \\ \&c. &= \&c. \end{aligned}\right\} \text{ from Art. 8 of this Chapter.}$$

$$\left.\begin{aligned} \sin\theta &= -\sin(\pi+\theta) \\ \cos\theta &= -\cos(\pi+\theta) \\ \tan\theta &= \tan(\pi+\theta) \\ \&c. &= \&c. \end{aligned}\right\} \text{ from Art. 9 of this Chapter.}$$

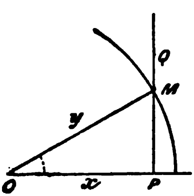
$$\left.\begin{aligned} \sin\theta &= -\cos\left(\frac{\pi}{2}+\theta\right) \\ \cos\theta &= \sin\left(\frac{\pi}{2}+\theta\right) \\ \tan\theta &= -\cot\left(\frac{\pi}{2}+\theta\right) \\ \&c. &= \&c. \end{aligned}\right\} \text{ from Art. 10 of this Chapter.}$$

$$\left.\begin{aligned} \sin\theta &= -\sin(2\pi-\theta) \\ \cos\theta &= \cos(2\pi-\theta) \\ \tan\theta &= -\tan(2\pi-\theta) \\ \&c. &= \&c. \end{aligned}\right\} \text{ from Art. 11 of this Chapter.}$$

$$\left.\begin{aligned} \sin\theta &= \sin(2\pi+\theta) \\ \cos\theta &= \cos(2\pi+\theta) \\ \tan\theta &= \tan(2\pi+\theta) \\ \&c. &= \&c. \end{aligned}\right\} \text{ from Art. 11 of this Chapter.}$$

$$\sin\theta = \sin(2n\pi+\theta) = -\sin(2n\pi-\theta)$$
$$\sin\theta = -\sin\{(2n+1)\pi+\theta\} = \sin\{(2n+1)\pi-\theta\}$$
$$\cos\theta = \cos(2n\pi+\theta) = \cos(2n\pi-\theta)$$
$$\cos\theta = -\cos\{(2n+1)\pi+\theta\} = -\cos\{(2n+1)\pi-\theta\}$$
$$\tan\theta = \tan(2n\pi+\theta) = -\tan(2n\pi-\theta)$$
$$\tan\theta = \tan\{(2n+1)\pi+\theta\} = -\tan\{(2n+1)\pi-\theta\}$$
$$\&c. = \&c. = \&c.$$

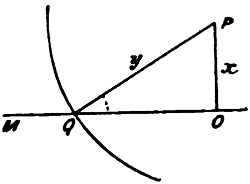
15. To construct an angle having a given sine, cosine, tangent, &c.



Let $\cos a = \dfrac{x}{y}$; find it. Take any line $OP = x$. Draw PQ perpendicular to it. With centre O and radius equal to y, describe an arc, cutting PQ in M; join OM, then

$\cos \angle POM = \dfrac{x}{y}$. $\therefore$ POM is the required angle.

Let $\sin a = \frac{x}{y}$; find it. Take any line OM, draw OP perpendicular to it and equal to x.



With centre P and radius equal to y, describe an arc, cutting MO in Q; join PQ, then

$$\sin PQO = \frac{x}{y}$$

$\therefore$ PQO is the required angle.

Let $\operatorname{cosec} a = \frac{x}{y}$; find it.

Take any indefinite line MO, as in the above figure. Draw OP perpendicular to it and equal to y, with P as centre, and radius equal to x, describe an arc, cutting OM in Q; join PQ;

$$\text{then } \operatorname{cosec} PQO = \frac{x}{y}$$

$\therefore$ PQO is the required angle.

Let $\tan a = \frac{x}{y}$; find it.

Take any line $OQ = y$, make OP perpendicular to it and equal to x; join PQ,

$$\text{then } \tan PQO = \frac{x}{y}$$

$\therefore$ PQO is the required angle.

Similarly for all the other functions.

Exercises.

1. What is the greatest value the *cosine* can have? what the *versine*? Can the *sine* ever be equal to or greater than unity?

2. What do you mean by the *complement* and *supplement* of an angle. What is the *complement* of 79° 57′ 45″? what the *supplement*?

3. Prove by figure that the "sine of an angle is equal to the cosine of its complement;"

$$\text{or that } \sin \alpha = \cos\left(\frac{\pi}{2} - \alpha\right)$$

$$\text{Also } \sin \alpha = \sin(\pi - \alpha)$$

4. Find the trigonometrical ratios of 45°, 30°, and 60°.

5. Prove the following by figure:—

$$\cos \alpha = -\cos(\pi + \alpha)$$

$$\sin \alpha = -\cos\left(\frac{\pi}{2} + \alpha\right)$$

$$\sin \alpha = \sin(n\pi - \alpha), \text{ where } n \text{ is any odd integer.}$$

6. The secant of an angle $= \dfrac{2}{\sqrt{3}}$; construct it.

7. Give the arithmetical value and sign of the product of $\sin A . \cos A$ for every multiple of 30° up to 360°, beginning from 0°.

8. Trace the changes of sign of the tangent of an angle from 0° to 360°. Give the numerical values of the tangents of every $22\frac{1}{2}$° from 0° to 360°.

9. Find the arithmetical values of the tangents of 15°, 75°, 105°, 165°, 195°, 255°, 285°, 345°, − 1000°.

10. Trace the change in magnitude and sign of the secant of an angle through 180°.

11. Show that $\sin(-\alpha) = -\sin \alpha$
$\tan(-\alpha) = -\tan \alpha.$

12. Write down the values of the sines 210°, − 120°, 165°.

13. Trace the changes in sign and value of $\cos A - \sin A$ from 0° to 360°.

14. Find tan 1050°.

Here (Art. 14)
$$\tan 1050° = \tan(2 . 360° + 330°)$$
$$= \tan 330° = \tan(360° - 30°)$$
$$- \tan 30° = -\frac{1}{\sqrt{3}}$$
$$\text{or } \tan 1050° = \tan(6 . 180° - 30°) = -\tan 30°$$

15. Find all the values of α up to 360° which satisfy the equation,

$$\sin 5\alpha . \cos 3\alpha = \sin 4\alpha . \cos 4\alpha .$$

16. Prove that the expression $2n\pi \pm \alpha$ is one that includes all angles having the same *cosine* as α, where n is any integer, positive or negative.

17. Prove that the formula $n\pi + (-1)^n\alpha$ includes all angles having the same *sine* as α, where n is as in the last problem.

18. Show that the formula $n\pi + \alpha$ includes all angles having the same *tangent* as α, where n is as before.

19. Write down the general value of θ if $\cos\theta = -\frac{1}{2}$.

20. Give the general value of α if $\tan\alpha = \pm\frac{1}{\sqrt{3}}$

21. Find the general value of α if $\cos 3\alpha = -\frac{1}{2}$.

22. If α and β be the least angles which satisfy the equations,

$$\tan\alpha = \frac{a}{b} \qquad \sin\beta = \frac{\sqrt{a^2+b^2}}{2}$$

Show that the formula $n\pi + (-1)^n\beta - \alpha$ contains all the values of θ which satisfy the equations,

$$\sin\theta + \sin\phi = b$$
$$\cos\theta + \cos\phi = a$$

23. Solve the equation,

$$\tan\alpha + 3\cot\alpha = 4.$$

24. Find the sine of $\left\{(2n \pm 3)\frac{\pi}{2} + \beta\right\}$ where n is any integer.

25. Find the tangent of $\left(\frac{3\pi}{2} + \alpha\right)$ in its most simple form.

26. Prove that

$$\cos n\left(\frac{\pi}{2} - \alpha\right) = (-1)^{\frac{n-1}{2}} \sin n\alpha$$

where n is any odd integer.

27. If $\cot(\pi\tan\theta) = \tan(\pi . \cot\theta)$, find $\tan\theta$.

28. Find $\sec\left\{(6n+3)\frac{\pi}{2} + \alpha\right\}$ if $\sin\alpha = a$.

29. Given $2\sin A = \tan A$, find $\sin A$, $\cos A$, &c.

30. If $\sin A . \cos A = \frac{m}{\sqrt{1+m^2}}$ find $\sin A$, $\cos A$, &c.

31. If $\sin A . \sec B = m$
$\cos A . \operatorname{cosec} B = n$, find $\sec A$, $\sec B$.

32. Prove the following:—

(a) $\sin\left(\frac{3\pi}{2} \pm A\right) = -\cos A$

(b) $\cos\left(\frac{3\pi}{2} \pm A\right) = \pm\sin A$

(c) $\sin\left\{(4n+1)\frac{\pi}{2} \pm A\right\} = \cos A$

(d) $\cos\left\{(4n+1)\frac{\pi}{2} \pm A\right\} = \pm\sin A$

$$(e)\ \tan\left\{(4n+3)\frac{\pi}{2} \pm A\right\} = \pm \cot A$$

$$(f)\ \cot\left\{(4n+3)\frac{\pi}{2} \pm A\right\} = \pm \tan A$$

$$(g)\ \sin\left\{(3n+1)\frac{\pi}{2} \pm A\right\} = \pm \cos A$$

$$(h)\ \cos\left\{(3n+1)\frac{\pi}{2} \pm A\right\} = \pm \sin A$$

$$(k)\ \tan\left\{(3n-1)\frac{\pi}{2} \pm A\right\} = \pm \cot A$$

$$(m)\ \cot\left\{(3n-1)\frac{\pi}{2} \pm A\right\} = \pm \tan A$$

33. Prove that

$$\tan A = \frac{\cos A}{\sin A \,.\, \cot^2 A}$$

$$\cot A = \frac{\sin A}{\cos A \,.\, \tan^2 A}$$

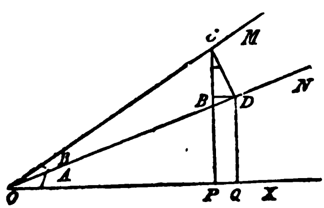
CHAPTER V.

TRIGONOMETRICAL FUNCTIONS OF TWO OR MORE ANGLES.

1. To prove

$$\sin(A+B) = \sin A \,.\, \cos B + \cos A \,.\, \sin B$$

and $\cos(A+B) = \cos A \,.\, \cos B - \sin A \,.\, \sin B.$



Make any two angles XON, NOM equal to A and B respectively. Take any point C . in OM and let fall perpendiculars CD, CP on ON, OX; and DB perpendicular to CP.

Now since $\angle$ CBD is a right angle,

$\therefore$ $\angle$s BCD, BDC make a right angle,

and are consequently equal to $\angle$ CDO.

Take away the common angle CDB,

$\therefore$ $\angle$ BCD = $\angle$ BDO,

but $\angle$ BDO = $\angle$ A, being alternate angles;

hence $\angle$ BCD = $\angle$ A.

$$\text{Now } \sin(A+B) = \frac{CP}{CO} = \frac{BP+BC}{CO} = \frac{DQ}{CO} + \frac{BC}{CO}$$

$$= \frac{DQ}{DO} \cdot \frac{DO}{CO} + \frac{BC}{CD} \cdot \frac{CD}{CO}$$

$$= \sin A \,.\, \cos B + \cos A \,.\, \sin B.$$

Again,

$$\cos(A+B) = \frac{OP}{OC} = \frac{OQ - PQ}{OC} = \frac{OQ}{OC} - \frac{BD}{OC}\,^{*}$$

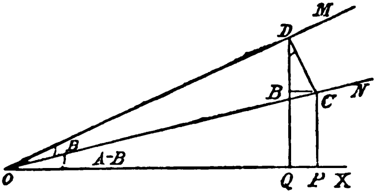
$$= \frac{OQ}{OD} \cdot \frac{OD}{OC} - \frac{BD}{CD} \cdot \frac{CD}{OC}$$

$$= \cos A \,.\, \cos B - \sin A \,.\, \sin B.$$

2. To prove that

$$\sin(A-B) = \sin A \,.\, \cos B - \cos A \,.\, \sin B$$

and $\cos(A-B) = \cos A \,.\, \cos B + \sin A \,.\, \sin B.$



Let XOM and MON be any angles A and B, then $\angle$ XON = A − B. Take any point C in ON and let fall CD, CP perpendicular to OM and OX respectively, and CB perpendicular to DQ.

* In this and the corresponding part of the last proposition the student may have some difficulty at first in understanding how we follow from this point. We simply divide the numerator and

$$\text{Now } \sin(A - B) = \frac{CP}{CO} = \frac{BQ}{CO} = \frac{DQ}{CO} - \frac{DB}{CO}$$
$$= \frac{DQ}{DO} \cdot \frac{DO}{CO} - \frac{DB}{DC} \cdot \frac{DC}{CO}$$
$$= \sin A \cdot \cos B - \cos A \cdot \sin B.$$

It can be seen that the angle CDB $= \angle$ A, for since $\angle$ DQO is a right angle and $\angle$ CDO is another, hence $\angle s$ DOQ, ODQ $= \angle$ ODC; take away the common angle ODQ and $\angle$ CDB $= \angle$ DOQ $= \angle$ A.

$$\text{Again, } \cos(A - B) = \frac{OP}{OC} = \frac{OQ + QP}{OC} = \frac{OQ}{OC} + \frac{BC}{OC}$$
$$= \frac{OQ}{OD} \cdot \frac{OD}{OC} + \frac{BC}{CD} \cdot \frac{CD}{OC}$$
$$= \cos A \cdot \cos B + \sin A \cdot \sin B.$$

These four formulæ should be committed to memory thus:—

sin of sum of two angles = sin 1^{st} cos 2^{nd} + cos 1^{st} sin 2^{nd}
sin of diff. of two angles = sin 1^{st} cos 2^{nd} − cos 1^{st} sin 2^{nd}
cos of sum of two angles = product of cosines − product of sines.
cos of diff. of two angles = product of cosines + product of sines.

3. In the figure here shown, A and B are supposed to be each less than a right angle. So also is their sum and difference. The propositions may be proved where

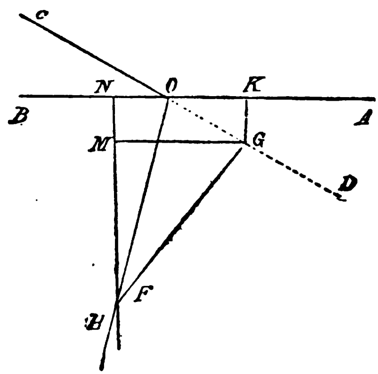
---

denominator of each fraction by the same line, which will in no way alter the value of our original fraction;

$$\text{for } \frac{OQ}{OC} = \frac{\frac{OQ}{OD}}{\frac{OC}{OD}} = \frac{OQ}{OD} \cdot \frac{OD}{OC}$$

The others are got in precisely the same way; remembering always to divide by the *hypothenuse* of the right-angled triangle, of which this line under consideration is a side.

A and B are each less than a right angle, and their sum greater than a right angle, or where A and B are each *greater* than a right angle. We give the proof of the latter, and the student should make himself master of the other case by following a similar method of proof.



Let $\angle$ s AOC, COH be equal to A and B respectively. Take any point F in the line OH and let fall perpendiculars FG, FN, GM, GK on OC produced, AB, NF, AB respectively. Let $\angle$ KOG $= \alpha$.

$$\text{Now } \sin(A + B) = -\frac{NF}{OF} = -\frac{KG}{OF} - \frac{MF}{OF}$$

$$= -\frac{KG}{OG} \cdot \frac{OG}{OF} - \frac{MF}{FG} \cdot \frac{FG}{OF}$$

$= \sin(360° - \alpha) . \cos . \angle GOF - \cos(360° - \alpha) . \sin \angle GOF$
$= - \sin \alpha . \cos \angle GOF - \cos \alpha . \sin \angle GOF$
$= - \sin \angle COA \times - \cos \angle COF - \cos \alpha . \sin \angle COF$
$= \sin A . \cos B + \cos A . \sin B.$

4. Collecting the four formulæ, we have now

$\sin(A + B) = \sin A . \cos B + \cos A . \sin B$ . . . 1
$\sin(A - B) = \sin A . \cos B - \cos A . \sin B$ . . . 2
$\cos(A + B) = \cos A . \cos B - \sin A . \sin B$ . . . 3
$\cos(A - B) = \cos A . \cos B + \sin A . \sin B$ . . . 4

By adding and subtracting 1 and 2, and 3 and 4 respectively, we get—

$$\sin(A + B) + \sin(A - B) = 2 \sin A . \cos B$$
$$\sin(A + B) - \sin(A - B) = 2 \cos A . \sin B$$
$$\cos(A + B) + \cos(A - B) = 2 \cos A . \cos B$$
$$\cos(A - B) - \cos(A + B) = 2 \sin A . \sin B$$

Now let $A + B = \alpha$
$A - B = \beta$

$$\therefore\ A = \frac{\alpha + \beta}{2} \text{ and } B = \frac{\alpha - \beta}{2}$$

Substituting these values of A, B, A + B and A − B in the four last formulæ, we have then,

$$\sin\alpha + \sin\beta = 2 \sin\frac{\alpha + \beta}{2} . \cos\frac{\alpha - \beta}{2} \quad \ldots . 5$$

$$\sin\alpha - \sin\beta = 2 \cos\frac{\alpha - \beta}{2} . \sin\frac{\alpha - \beta}{2} \quad \ldots . 6$$

$$\cos\alpha + \cos\beta = 2 \cos\frac{\alpha + \beta}{2} . \cos\frac{\alpha - \beta}{2} \quad \ldots . 7$$

$$\cos\beta - \cos\alpha = 2 \sin\frac{\alpha + \beta}{2} . \sin\frac{\alpha - \beta}{2} \quad \ldots . 8$$

These might thus be read:—

The sum of the sines of two angles = twice sin half sum × cos half diff.

The diff. " " = twice cos half sum × sin half diff.

The sum of the cosines of two angles = twice cos half sum × cos half diff.

The diff. " " = twice sin half sum × sin half diff.

In this latter case it must be borne in mind that the cosine of the *lesser* angle comes first.

5. In equation (1), Art. 4, and also in equation (3), let B = A.

$$\text{Hence } \sin 2A = 2 \sin A \,.\, \cos A \; \ldots\ldots \; 9$$
$$\cos 2A = \cos^2 A - \sin^2 A \ldots\ldots 10$$
$$= 1 - \sin^2 A - \sin^2 A$$
$$= 1 - 2 \sin^2 A \; \ldots\ldots \; 11$$
$$\text{Or, again, } \cos 2A = \cos^2 A - \sin^2 A$$
$$= \cos^2 A - (1 - \cos^2 A)$$
$$= 2 \cos^2 A - 1 \; \ldots\ldots \; 12$$

Now if instead of $2 \cdot A$ on the left-hand side of each of these four results we write A, we must, on the other side, replace A by $\frac{A}{2}$.

$$\therefore \sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2} \; \ldots\ldots \; 13$$
$$\cos A = \cos^2 \frac{A}{2} - \sin^2 \frac{A}{2} \ldots\ldots 14$$
$$= 1 - 2 \sin^2 \frac{A}{2} \; \ldots\ldots\ldots \; 15$$
$$= 2 \cos^2 \frac{A}{2} - 1 \; \ldots\ldots\ldots \; 16$$

By transposition in equations (11) and (12), we get

$$1 + \cos 2A = 2 \cos^2 A \text{ or } 1 + \cos A = 2 \cos^2 \frac{A}{2}$$
$$1 - \cos 2A = 2 \sin^2 A \text{ ,, } 1 - \cos A = 2 \sin^2 \frac{A}{2}$$

6. To find an expression for tan (A + B), and tan (A − B).

$$\text{Tan}\,(A + B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin A \,.\, \cos B + \cos A \,.\, \sin B}{\cos A \,.\, \cos B - \sin A \,.\, \sin B}$$

$$= \frac{\dfrac{\sin A \,.\, \cos B}{\cos A \,.\, \cos B} + \dfrac{\cos A \,.\, \sin B}{\cos A \,.\, \cos B}}{1 - \dfrac{\sin A \,.\, \sin B}{\cos A \,.\, \cos B}}$$ By dividing numerator and denominator by cos A . cos B.

$$= \frac{\tan A + \tan B}{1 - \tan A \,.\, \tan B} \; \ldots\ldots \; 17$$

Let B = A.

$$\therefore \tan 2A = \frac{2 \tan A}{1 - \tan^2 A} \quad \ldots\ldots 18$$

$$\text{Or } \tan A = \frac{2 \tan \frac{A}{2}}{1 - \tan^2 \frac{A}{2}} \quad \ldots\ldots 19$$

Similarly,

$$\tan (A - B) = \frac{\tan A - \tan B}{1 + \tan A \cdot \tan B} \quad \ldots 20$$

7. $\text{Cot}(A + B) = \dfrac{1}{\tan (A + B)} = \dfrac{1 - \tan A \cdot \tan B}{\tan A + \tan B}$

$$= \frac{1 - \dfrac{1}{\cot A \cdot \cot B}}{\dfrac{1}{\cot A} + \dfrac{1}{\cot B}} = \frac{\cot A \cdot \cot B - 1}{\cot A + \cot B}$$

Let B = A.

$$\therefore \cot 2A = \frac{\cot^2 A - 1}{2 \cot A}$$

$$\text{or } \cot A = \frac{\cot^2 \frac{A}{2} - 1}{2 \cot \frac{A}{2}}$$

8. These expressions can be got geometrically also. See figure to Art. 1 of this chapter.

$$\tan (A + B) = \frac{CP}{PO} = \frac{BP + BC}{PO} = \frac{DQ + BC}{PO}$$

$$= \frac{DQ + CB}{OQ - PQ} = \frac{\dfrac{DQ}{OQ} + \dfrac{CB}{OQ}}{1 - \dfrac{PQ}{OQ}}$$

$$= \frac{\frac{DQ}{OQ} + \frac{CB}{OQ}}{1 - \frac{BD}{OQ}} = \frac{\frac{DQ}{OQ} + \frac{CB}{OQ}}{1 - \frac{BD}{BC} \cdot \frac{BC}{OQ}}$$

$$= \frac{\tan A + \tan B}{1 - \tan A \,.\, \tan B} \qquad \text{since } \frac{CB}{OQ} = \frac{CD}{DO} \text{ by similar triangles.}$$

9. The formula $\cos 2A = \cos^2 A - \sin^2 A$ can also be proved geometrically.

In the same figure as last, let $\angle A = \angle B$, then

$$\cos 2A = \frac{OP}{OC} = \frac{OQ}{AF} - \frac{BD}{AF}$$

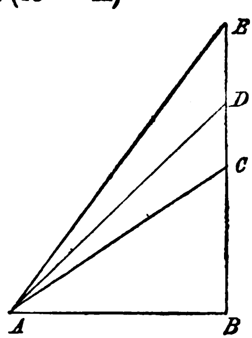
$$= \frac{OQ}{OD} \cdot \frac{OD}{AF} - \frac{BD}{DC} \cdot \frac{DC}{AF}$$

$$= \cos A \,.\, \cos A - \sin A \,.\, \sin A$$

$$= \cos^2 A - \sin^2 A.$$

10. Prove geometrically that

$$\tan (45° + A) = \frac{1}{\tan (45° - A)}$$



Let $\angle BAD = 45°$

Make $\angle DAE = DAC = A$

Now $\angle ACB = \angle ADB + \angle CAD$

$= \angle BAD + \angle DAE$

$= \angle BAE$

And $\angle B$ is common to the two triangles BAC, BAE.

$\therefore$ by similar triangles,

$$\frac{EB}{AB} = \frac{AB}{BC}$$

or $\tan (45° + A) = \cot (45° - A)$

$$= \frac{1}{\tan (45° - A)}$$

11. $$\tan(A + 45°) = \frac{\tan A + \tan 45°}{1 - \tan A . \tan 45°}$$

$$= \frac{1 + \tan A}{1 - \tan A}$$

12. To find expressions for sin 3 A, cos 3 A, tan 3 A.

$$\begin{aligned}
\sin 3A &= \sin(2A + A) \\
&= \sin 2A . \cos A + \cos 2A . \sin A \\
&= 2 \sin A . \cos A . \cos A + (1 - 2\sin^2 A) \sin A \\
&= 2 \sin A \cos^2 A + (1 - 2\sin^2 A) \sin A \\
&= 2 \sin A (1 - \sin^2 A) + (1 - 2\sin^2 A) \sin A \\
&= 2 \sin A - 2\sin^3 A + \sin A - 2\sin^3 A \\
&= 3 \sin A - 4 \sin^3 A
\end{aligned}$$

$$\begin{aligned}
\cos 3A &= \cos(2A + A) \\
&= \cos 2A . \cos A - \sin 2A . \sin A \\
&= (2\cos^2 A - 1) \cos A - 2 \sin A . \cos A . \sin A \\
&= 2\cos^3 A - \cos A - 2\sin^2 A . \cos A \\
&= 2\cos^3 A - \cos A - 2\cos A (1 - \cos^2 A) \\
&= 4\cos^3 A - 3\cos A
\end{aligned}$$

$$\tan 3A = \tan(2A + A)$$

$$= \frac{\tan 2A + \tan A}{1 - \tan 2A . \tan A} = \frac{\dfrac{2\tan A}{1 - \tan^2 A} + \tan A}{1 - \dfrac{2\tan A}{1 - \tan^2 A} . \tan A}$$

$$= \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$$

13. To prove

$$\frac{\sin A + \sin B}{\sin A - \sin B} = \frac{\tan \dfrac{A + B}{2}}{\tan \dfrac{A - B}{2}}$$

Now

$$\frac{\sin A + \sin B}{\sin A - \sin B} = \frac{2 \sin \dfrac{A + B}{2} . \cos \dfrac{A - B}{2}}{2 \cos \dfrac{A + B}{2} . \sin \dfrac{A - B}{2}}$$

$$= \tan\frac{A + B}{2} \cdot \cot\frac{A - B}{2}$$

$$= \frac{\tan\frac{A + B}{2}}{\tan\frac{A - B}{2}}$$

14. Prove that

$$\frac{\tan A + \tan B}{\tan A - \tan B} = \frac{\sin (A + B)}{\sin (A - B)}$$

Now

$$\frac{\tan A + \tan B}{\tan A - \tan B} = \frac{\frac{\sin A}{\cos A} + \frac{\sin B}{\cos B}}{\frac{\sin A}{\cos A} - \frac{\sin B}{\cos B}}$$

$$= \frac{\sin A \cdot \cos B + \cos A \cdot \sin B}{\sin A \cdot \cos B - \cos A \cdot \sin B}$$

$$= \frac{\sin (A + B)}{\sin (A - B)}$$

15. We might get other expressions for sin 2 A and cos 2 A besides those already found in Art. 5.

$$\sin 2A = 2 \sin A \cdot \cos A$$

$$= 2\frac{\sin A}{\cos A} \cdot \cos^2 A$$

$$= 2 \tan A \cdot \frac{1}{1 + \tan^2 A} = \frac{2 \tan A}{1 + \tan^2 A}$$

$$\cos 2A = 2 \cos^2 A - 1$$

$$= 2 \cdot \frac{1}{1 + \tan^2 A} - 1$$

$$= \frac{2 - 1 - \tan^2 A}{1 + \tan^2 A} = \frac{1 - \tan^2 A}{1 + \tan^2 A}$$

16. To find expressions for sine, cosine, &c., of (A + B + C).

Let A + B be considered as one angle—say α;

$$\therefore \sin(A + B + C) = \sin(\alpha + C)$$
$$= \sin\alpha \cdot \cos C + \cos\alpha \cdot \sin C$$
$$= \sin(A + B) \cdot \cos C + \cos(A + B) \cdot \sin C$$
$$= (\sin A \cdot \cos B + \cos A \cdot \sin B)\cos C + (\cos A \cdot \cos B - \sin A \cdot \sin B)\sin C$$
$$= \sin A \cdot \cos B \cdot \cos C + \sin B \cdot \cos A \cdot \cos C + \sin C \cdot \cos A \cdot \cos B - \sin A \cdot \sin B \cdot \sin C.$$

A similar expression may be got for $\cos(A + B + C)$

$$\tan(A + B + C) = \frac{\tan(A + B) + \tan C}{1 - \tan(A + B) \cdot \tan C}$$

$$= \frac{\dfrac{\tan A + \tan B}{1 - \tan A \cdot \tan B} + \tan C}{1 - \dfrac{\tan A + \tan B}{1 - \tan A \cdot \tan B} \cdot \tan C}$$

$$= \frac{\tan A + \tan B + \tan C - \tan A \cdot \tan B \cdot \tan C}{1 - \tan A \cdot \tan B - \tan A \cdot \tan C - \tan B \cdot \tan C}$$

If $A + B + C = 180°$

and since $\tan 180° = 0$

$$\therefore \tan A + \tan B + \tan C = \tan A \cdot \tan B \cdot \tan C$$

or since $\cot(A + B + C)$

$$= \frac{1 - \tan A \cdot \tan B - \tan A \cdot \tan C - \tan B \cdot \tan C}{\tan A + \tan B + \tan C - \tan A \cdot \tan B \cdot \tan C}$$

and since $\cot 90° = 0$

$\therefore$ if $A + B + C = 90°$, we have

$$\tan A + \tan B + \tan C = \tan A \cdot \tan B \cdot \tan C.$$

17. To find the trigonometrical functions of 15°, 75° 18°, 36°, 54°, 72°.

Since $\cos 2A + 1 = 2\cos^2 A$

$$\therefore \cos 30° + 1 = 2\cos^2 15°$$

$$\therefore \cos^2 15° = \frac{1 + \dfrac{\sqrt{3}}{2}}{2} = \frac{\sqrt{3} + 2}{4}$$

$$\therefore \cos 15 = \frac{\sqrt{\sqrt{3} + 2}}{2} = \frac{\sqrt{3} + 1}{2\sqrt{2}}$$

Or it may be found in, perhaps, the following simpler manner :—

$$\sin 15^\circ = \sin (45^\circ - 30^\circ)$$
$$= \sin 45^\circ \cos 30^\circ - \cos 45^\circ . \sin 30^\circ$$
$$= \frac{1}{\sqrt{2}} . \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} . \frac{1}{2}$$
$$= \frac{\sqrt{3} - 1}{2\sqrt{2}}$$

$$\cos 15^\circ = \cos (45^\circ - 30^\circ)$$
$$= \cos 45^\circ . \cos 30^\circ + \sin 45^\circ \sin 30^\circ$$
$$= \frac{1}{\sqrt{2}} . \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} . \frac{1}{2}$$
$$= \frac{\sqrt{3} + 1}{2\sqrt{2}}$$

$$\tan 15^\circ = \frac{\sin 15^\circ}{\cos 15^\circ} = \frac{\sqrt{3} - 1}{\sqrt{3} + 1} = 2 - \sqrt{3}$$

$$\cot 15^\circ = \frac{\cos 15^\circ}{\sin 15^\circ} = \frac{\sqrt{3} + 1}{\sqrt{3} - 1} = 2 + \sqrt{3}$$

$$\sec 15^\circ = \frac{1}{\cos 15^\circ} = \frac{2\sqrt{2}}{\sqrt{3} + 1}$$

$$\operatorname{cosec} 15^\circ = \frac{1}{\sin 15^\circ} = \frac{2\sqrt{2}}{\sqrt{3} - 1}$$

since 15° and 75° are complementary angles ;

$$\therefore \sin 15^\circ = \cos 75^\circ = \frac{\sqrt{3} - 1}{2\sqrt{2}}$$
$$\cos 15^\circ = \sin 75^\circ = \frac{\sqrt{3} + 1}{2\sqrt{2}}$$
$$\tan 15^\circ = \cot 75^\circ = 2 - \sqrt{3}$$
$$\cot 15^\circ = \tan 75^\circ = 2 + \sqrt{3}$$
$$\sec 15^\circ = \operatorname{cosec} 75^\circ = \frac{2\sqrt{2}}{\sqrt{2} + 1}$$
$$\operatorname{cosec} 15^\circ = \sec 75^\circ = \frac{2\sqrt{2}}{\sqrt{3} - 1}$$

18. Now let $A = 18^\circ \therefore 2A = 36^\circ$ and $3A = 54^\circ$

$$\text{but } \sin 36^\circ = \cos 54^\circ$$
$$\therefore \sin 2A = \cos 3A$$
$$\therefore 2 \sin A . \cos A = 4 \cos^3 A - 3 \cos A$$
$$2 \sin A = 4 \cos^2 A - 3$$
$$= 4 (1 - \sin^2 A) - 3$$
$$\therefore 4 \sin^2 A + 2 \sin A = 1$$
$$\therefore \sin A = \pm \frac{\sqrt{5} - 1}{4}$$

We neglect the negative sign, since 18° is in the 1st quadrant, and therefore positive.

$$\text{Hence } \sin 18^\circ = \frac{\sqrt{5} - 1}{4}$$

$$\cos 18^\circ = \sqrt{1 - \left(\frac{\sqrt{5} - 1}{4}\right)^2} = \frac{\sqrt{10 + 2\sqrt{5}}}{4}$$

$$\tan 18^\circ = \frac{\sin 18^\circ}{\cos 18^\circ} = \frac{\sqrt{5} - 1}{\sqrt{10 + 2\sqrt{5}}}$$

Similar expressions can be got for the other functions of 18°.

$$\therefore \sin 18^\circ = \cos 72^\circ = \frac{\sqrt{5} - 1}{4}$$

$$\cos 18^\circ = \sin 72^\circ = \frac{\sqrt{10 + 2\sqrt{5}}}{4}$$

$$\tan 18^\circ = \cot 72^\circ = \frac{\sqrt{5} - 1}{\sqrt{10 + 2\sqrt{5}}}$$

$$\&c. = \&c. = \&c.$$

19. Again, since

$$1 + \cos 2A = 2 \sin^2 A$$
$$\therefore 1 - \cos 36^\circ = 2 \sin^2 18^\circ$$
$$\therefore \cos 36^\circ = 1 - 2 \sin^2 18^\circ$$
$$= 1 - 2 . \left(\frac{\sqrt{5} - 1}{4}\right)^2$$
$$= 1 - 2 . \left(\frac{6 - 2\sqrt{5}}{16}\right)$$

$$= 1 - \frac{6 - 2\sqrt{5}}{8} = \frac{2 + 2\sqrt{5}}{8} = \frac{\sqrt{5} + 1}{4}$$

$$\sin 36^\circ = \sqrt{1 - \cos^2 36^\circ}$$

$$= \sqrt{1 - \frac{6 + 2\sqrt{5}}{16}} = \frac{\sqrt{10 - 2\sqrt{5}}}{4}$$

$$\tan 36^\circ = \frac{\sin 36^\circ}{\cos 36^\circ} = \frac{\sqrt{10 - 2\sqrt{5}}}{\sqrt{5} + 1}$$

Similar expressions can be got for the other functions of 36°.

$$\therefore \sin 36^\circ = \cos 54^\circ = \frac{\sqrt{10 - 2\sqrt{5}}}{4}$$

$$\cos 36^\circ = \sin 54^\circ = \frac{\sqrt{5} + 1}{4}$$

$$\tan 36^\circ = \cot 54^\circ = \frac{\sqrt{10 - 2\sqrt{5}}}{\sqrt{5} + 1}$$

&c. = &c. = &c.

The student should work out the preceding values of these angles again and again, until the method and results are well fixed in his memory.

Exercises.

1. Establish the formulæ:

$$\sin (A + B) = \sin A \,.\, \cos B + \cos A \,.\, \sin B$$
$$\cos (A + B) = \cos A \,.\, \cos B - \sin A \,.\, \sin B$$

drawing the figure for the case in which A and B are each greater than a right angle, and (A + B) between two and three right angles.

2. Deduce the expression for cos (A − B) and cos 3 A from that of sin (A + B).

3. Prove the following formulæ:—

$$\sin \alpha + \sin \beta = 2 \sin \frac{\alpha + \beta}{2} \,.\, \cos \frac{\alpha - \beta}{2} \,.$$

$$\sin \alpha = 2 \sin \frac{\alpha}{2} \,.\, \cos \frac{\alpha}{2}$$

$$2 \cos^2 \alpha = 1 + \cos 2\alpha$$

$$\cos 2A = \cos^2 A - \sin^2 A.$$

4. Prove the formula for tan (A + B) by figure, and thence deduce cot (A + B).

5. Prove the following formulæ:—

$$\tan A = \frac{2 \tan \frac{A}{2}}{1 - \tan^2 \frac{A}{2}}$$

$$\tan 3A = \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}$$

$$\cos A = 4 \cos^3 \frac{A}{3} - 3 \cos \frac{A}{3}$$

$$\sin A = 3 \sin \frac{A}{3} - 4 \sin^3 \frac{A}{3}$$

6. Find an expression for tan (A + B + C), and from it deduce tan 3 A.

7. Prove that $\dfrac{\tan \alpha + \tan \beta}{\tan \alpha - \tan \beta} = \dfrac{\sin (\alpha + \beta)}{\sin (\alpha - \beta)}$

8. Find tan 36°.

The following six questions will be worked by way of illustration.

9. Eliminate α from the equations,

$$\left.\begin{array}{r} \operatorname{cosec}^2 \alpha = a \tan \alpha \\ \sec^2 \alpha = b \cot \alpha \end{array}\right\} \begin{array}{l} \ldots\ldots 1 \\ \ldots\ldots 2 \end{array}$$

By multiplication we get

$$\frac{1}{\sin^2 \alpha \,.\, \cos^2 \alpha} = ab$$

$$\therefore \frac{1}{\sin \alpha \,.\, \cos \alpha} = \sqrt{ab}\,.\ \text{hence}$$

$$\sin \alpha \,.\, \cos \alpha = \frac{1}{\sqrt{ab}} \quad \ldots\ldots\ldots 3$$

Now from (1) we have

$$a \sin^3 \alpha = \cos \alpha$$

$$\therefore a \sin^4 \alpha = \sin \alpha \,.\, \cos \alpha \ldots\ldots 4$$

and from (2) we have

$$b \cos^3 \alpha = \sin \alpha$$

$$\therefore b \cos^4 \alpha = \sin \alpha \,.\, \cos \alpha \ldots\ldots 5$$

$$\text{Hence } \frac{1}{\sqrt{ab}} = a \sin^4 \alpha$$

$$\therefore \sin^4 \alpha = \frac{1}{a^{\frac{3}{2}}\, b^{\frac{1}{2}}}$$

$$\text{or } \sin^2 \alpha = \frac{1}{a^{\frac{3}{4}}\, b^{\frac{1}{4}}}$$

$$\text{Similarly } \frac{1}{\sqrt{ab}} = b \, . \cos^4 a$$

$$\therefore \cos^2 a = \frac{1}{a^{\frac{1}{4}} \, b^{\frac{3}{4}}}$$

and by addition,

$$1 = \frac{1}{a^{\frac{3}{4}} \, b^{\frac{1}{4}}} + \frac{1}{a^{\frac{1}{4}} \, b^{\frac{3}{4}}}$$

$$\text{or } ab = a^{\frac{1}{4}} \, b^{\frac{1}{4}} \left(a^{\frac{1}{2}} + b^{\frac{1}{2}}\right)$$

$$\text{or } a^{\frac{3}{4}} \, b^{\frac{3}{4}} = a^{\frac{1}{2}} + b^{\frac{1}{2}}$$

$$\therefore a^{\frac{3}{2}} \, b^{\frac{3}{2}} = \left(a^{\frac{1}{2}} + b^{\frac{1}{2}}\right)^2$$

$$\text{or } (ab)^{\frac{3}{2}} = \left(a^{\frac{1}{2}} + b^{\frac{1}{2}}\right)^2$$

10. If $\left.\begin{aligned} \sin\theta &= a \sin\phi \\ \tan\theta &= b \tan\phi \end{aligned}\right\}$ $\begin{aligned} &\ldots\ldots\, 1 \\ &\ldots\ldots\, 2, \end{aligned}$ find $\cos\theta$.

Now, dividing (1) by (2), we get at once

$$\cos\theta = \frac{a}{b} \cos\phi$$

$$\text{but } \cos\phi = \sqrt{1 - \sin^2\phi}$$

$$= \sqrt{1 - \frac{\sin^2\theta}{a^2}}$$

$$\therefore \cos\theta = \frac{a}{b} \sqrt{\frac{a^2 - \sin^2\theta}{a}}$$

Hence by squaring

$$b^2 \cos^2\theta = a^2 - \sin^2\theta$$

$$= a^2 - 1 + \cos^2\theta$$

$$\therefore \cos^2\theta \, (b^2 - 1) = a^2 - 1$$

$$\therefore \cos\theta = \sqrt{\frac{a^2 - 1}{b^2 - 1}}$$

11. Eliminate a from the equations,

$$\left.\begin{aligned} \frac{x}{a \sin a} + \frac{y}{b \cos a} &= 1 \\ \frac{x}{a \sin^2 a} + \frac{y}{b \cos^2 a} &= 0 \end{aligned}\right\} \begin{aligned} &\ldots\ldots\, (1) \\ &\ldots\ldots\, (2) \end{aligned}$$

from (2) $\dfrac{x}{a \sin^2 a} = - \dfrac{y}{b \cos^2 a}$ and from this

4. Prove the formula for tan (A + B) by figure, and thence deduce cot (A + B).

5. Prove the following formulæ:—

$$\tan A = \frac{2 \tan \frac{A}{2}}{1 - \tan^2 \frac{A}{2}}$$

$$\tan 3 A = \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}$$

$$\cos A = 4 \cos^3 \frac{A}{3} - 3 \cos \frac{A}{3}$$

$$\sin A = 3 \sin \frac{A}{3} - 4 \sin^3 \frac{A}{3}$$

6. Find an expression for tan (A + B + C), and from it deduce tan 3 A.

7. Prove that $\dfrac{\tan \alpha + \tan \beta}{\tan \alpha - \tan \beta} = \dfrac{\sin (\alpha + \beta)}{\sin (\alpha - \beta)}$

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The following six questions will be worked by way of illustration.

9. Eliminate α from the equations,

$$\left.\begin{aligned} \operatorname{cosec}^2 \alpha &= a \tan \alpha \\ \sec^2 \alpha &= b \cot \alpha \end{aligned}\right\} \begin{aligned} \ldots\ldots 1 \\ \ldots\ldots 2 \end{aligned}$$

By multiplication we get

$$\frac{1}{\sin^2 \alpha . \cos^2 \alpha} = ab$$

$$\therefore \frac{1}{\sin \alpha . \cos \alpha} = \sqrt{ab}\text{. hence}$$

$$\sin \alpha . \cos \alpha = \frac{1}{\sqrt{ab}} \quad \ldots\ldots\ldots 3$$

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and from (2) we have

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$$\therefore b \cos^4 \alpha = \sin \alpha . \cos \alpha \ldots\ldots 5$$

$$\text{Hence } \frac{1}{\sqrt{ab}} = a \sin^4 \alpha$$

$$\therefore \sin^4 \alpha = \frac{1}{a^{\frac{3}{2}} b^{\frac{1}{2}}}$$

$$\text{or } \sin^2 \alpha = \frac{1}{a^{\frac{3}{4}} b^{\frac{1}{4}}}$$

$$\text{Similarly } \frac{1}{\sqrt{ab}} = b \, . \cos^4 \alpha$$

$$\therefore \cos^2 \alpha = \frac{1}{a^{\frac{1}{4}} \, b^{\frac{3}{4}}}$$

and by addition,

$$1 = \frac{1}{a^{\frac{3}{4}} \, b^{\frac{1}{4}}} + \frac{1}{a^{\frac{1}{4}} \, b^{\frac{3}{4}}}$$

$$\text{or } ab = a^{\frac{1}{4}} \, b^{\frac{1}{4}} \left(a^{\frac{1}{2}} + b^{\frac{1}{2}}\right)$$

$$\text{or } a^{\frac{3}{4}} \, b^{\frac{3}{4}} = a^{\frac{1}{2}} + b^{\frac{1}{2}}$$

$$\therefore a^{\frac{3}{2}} \, b^{\frac{3}{2}} = \left(a^{\frac{1}{2}} + b^{\frac{1}{2}}\right)^2$$

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$$= \sqrt{1 - \frac{\sin^2\theta}{a^2}}$$

$$\therefore \cos\theta = \frac{a}{b} \sqrt{\frac{a^2 - \sin^2\theta}{a}}$$

Hence by squaring

$$b^2 \cos^2\theta = a^2 - \sin^2\theta$$

$$= a^2 - 1 + \cos^2\theta$$

$$\therefore \cos^2\theta \, (b^2 - 1) = a^2 - 1$$

$$\therefore \cos\theta = \sqrt{\frac{a^2 - 1}{b^2 - 1}}$$

11. Eliminate α from the equations,

$$\left.\begin{aligned} \frac{x}{a \sin\alpha} + \frac{y}{b \cos\alpha} &= 1 \\ \frac{x}{a \sin^2\alpha} + \frac{y}{b \cos^2\alpha} &= 0 \end{aligned}\right\} \begin{aligned} &\ldots\ldots (1) \\ &\ldots\ldots (2) \end{aligned}$$

from (2) $\dfrac{x}{a \sin^2\alpha} = -\dfrac{y}{b \cos^2\alpha}$ and from this

$$\text{we get } \sin\alpha = \sqrt{\frac{bx}{bx - ay}}$$

$$\therefore a\sin\alpha = a\sqrt{\frac{bx}{bx - ay}} \text{ and from (2) we get}$$

$$b\cos\alpha = b\sqrt{\frac{ay}{ay - bx}}$$

substitute these values in (1)

$$\therefore \frac{x}{a\sqrt{\dfrac{bx}{bx - ay}}} + \frac{y}{b\sqrt{\dfrac{ay}{ay - bx}}} = 1$$

$$\frac{x\sqrt{bx - ay}}{a\sqrt{bx}} + \frac{y\sqrt{ay - bx}}{b\sqrt{ay}} = 1$$

from this we get

$$\sqrt{bx}\sqrt{bx - ay} + \sqrt{ay}\sqrt{ay - bx} = ab$$

12. Prove that

$$chd\,(\alpha + \beta) + chd\,(\alpha - \beta) = chd\,\alpha \,.\, chd\,(\pi - \beta)$$

Now since chd . angle = 2 sin . half angle, hence

$$chd\,(\alpha + \beta) + chd\,(\alpha - \beta) = 2\sin\frac{\alpha + \beta}{2} + 2\sin\frac{\alpha - \beta}{2}$$
$$= 2\left\{2\sin\frac{\alpha}{2}.\cos\frac{\beta}{2}\right\}$$
$$= 2\left\{2\sin\frac{\alpha}{2}.\sin\frac{\pi - \beta}{2}\right\}$$
$$= 2\sin\frac{\alpha}{2}.\,2\sin\frac{\pi - \beta}{2}$$
$$= chd\,\alpha.\,chd\,(\pi - \beta)$$

13. Show that

$$\sin 3(\alpha - 15^\circ) = 4\cos(\alpha - 45^\circ).\cos(\alpha + 15^\circ).\sin(\alpha - 15^\circ)$$

The expression on the right-hand side

$$= 2\sin(\alpha - 15^\circ)\left\{2\cos(\alpha - 45).\cos(\alpha + 15^\circ)\right\}$$
$$= 2\sin(\alpha - 15^\circ)\left\{\cos(2\alpha - 30^\circ) + \cos 60^\circ\right\}$$
$$= 2\sin(\alpha - 15^\circ).\cos(2\alpha - 30^\circ) + 2\sin(\alpha - 15^\circ) \times \tfrac{1}{2}$$
$$= \sin(3\alpha - 45) - \sin(\alpha - 15^\circ) + \sin(\alpha - 15^\circ)$$
$$= \sin(3\alpha - 45^\circ) = \sin 3(\alpha - 15^\circ).$$

14. Prove that

$$\sin 2\alpha.\sin\alpha = \cos\alpha - \cos\alpha.\cos 2\alpha$$

$$\text{Now } \sin 2\alpha.\sin\alpha = 2\sin^2\alpha.\cos\alpha$$
$$= \cos\alpha(1 - \cos 2\alpha)$$
$$= \cos\alpha - \cos\alpha.\cos 2\alpha$$

15. Solve the equation,

$$\frac{\sec 2a}{\operatorname{cosec} 3a} = 1$$

$$\text{Hence } \sqrt{1 + \tan^2 2a} = \sqrt{1 + \cot^2 3a}$$

$$\text{or } \tan^2 2a = \cot^2 3a$$

$$\therefore \tan 2a = \cot 3a$$

$$\text{Hence } 3a = \frac{\pi}{2} - 2a$$

$$\text{or } 5a = \frac{\pi}{2}$$

$$\therefore \frac{5a}{2} = \frac{\pi}{4}$$

And since the general expression for all angles having the same tangent is $\tan A = (n\pi + A)$

$$\text{Hence } \frac{5a}{2} = n\pi + \frac{\pi}{4}$$

$$\therefore a = \frac{2}{5}\left(n\pi + \frac{\pi}{4}\right)$$

16. If $\cot^2 \beta = \cot (a + \beta) \cot (a - \beta)$

prove $\sin 2\beta = \sqrt{2} \,.\, \sin a$

17. Show that

$$\left\{\frac{1}{\cos a} - \tan a\right\}^2 \left\{\sin\frac{a}{2} + \cos\frac{a}{2}\right\}^2 = \text{coversin } a$$

18. Solve the equation,

$$\frac{\cos a}{\cos 5a} = \frac{\cos 7a}{\cos 3a}$$

19. Show that

$$\frac{\sin (a - \beta)}{\sin a \,.\, \sin \beta} + \frac{\sin (\beta - \gamma)}{\sin \beta \,.\, \sin \gamma} + \frac{\sin (\gamma - a)}{\sin \gamma \,.\, \sin a} = 0$$

20. If $\dfrac{x}{y} = \dfrac{\cos a}{\cos \beta}$ then $\tan \dfrac{a + \beta}{2} \,.\, \tan \dfrac{a - \beta}{2} = \dfrac{x - y}{x + y}$

21. Prove that

$$(\text{chord } a)^2 = 4\left\{1 - \text{vers}\frac{a}{2}\right\} \text{vers}\frac{a}{2}\left\{1 + \frac{1}{1 - \text{vers}\frac{a}{2}}\right\}$$

22. Prove that

$$\frac{\sin^2 \frac{3\theta}{2}}{\sin^2 \frac{\theta}{2}} = 1 + 8 \cos\theta \,.\, \cos^2 \frac{\theta}{2}$$

23. Show that

$$2^{m-1}\sin\frac{\alpha}{2^m} - 2^{m-2}\sin\frac{\alpha}{2^{m-1}} = 2^m\sin\frac{\alpha}{2^m}.\sin^2\frac{\alpha}{2^{m+1}}$$

24. If $\cos\alpha = \dfrac{a\cos\beta - b}{a - b\cos\beta}$, then

$$\cot\frac{\alpha}{2} : \cot\frac{\beta}{2} :: (a-b)^{\frac{1}{2}} : (a+b)^{\frac{1}{2}}$$

25. If $\dfrac{a}{b} = \dfrac{\sin 2B}{\sin 2A}$ and $\dfrac{a}{b} = \dfrac{\sin B}{\sin A}$

$$\text{then } \sin A = \pm a\sqrt{\frac{a^2+b^2}{a^4+a^2b^2+b^4}}$$

26. If $\dfrac{\tan^2\alpha - 1}{2} = \tan^2\beta$, prove that

$$\frac{\cos 2\beta - 1}{2} = \cos 2\alpha$$

27. Eliminate α and β from the equations,

$$\tan\alpha = \frac{a}{b-c}, \tan\beta = \frac{a}{b+c}, \tan\frac{\alpha}{2} = n\tan\frac{\beta}{2}$$

28. If $\sin(\pi\cos\theta) = \cos(\pi\sin\theta)$, find $\sin 2\theta$.

29. Solve the equation,

$$3\tan\alpha.\tan 3\alpha = -1$$

30. Prove that if α, β, γ be the angles of a triangle,

$$\sin^3\alpha + \sin^3\beta + \sin^3\gamma = 3\cos\frac{\alpha}{2}.\cos\frac{\beta}{2}.\cos\frac{\gamma}{2} + \cos\frac{3\alpha}{2}.\cos\frac{3\beta}{2}\cos\frac{3\gamma}{2}$$

31. Prove that

$$\sin\alpha + \sin(72^\circ + \alpha) + \sin(36^\circ - \alpha) = \sin(72^\circ - \alpha) + \sin(36^\circ + \alpha)$$

32. If $\cos B(m + \cos A) = 1 + m\cos A$, show that

$$\tan\frac{A}{2} + \tan\frac{B}{2} = m\left(\tan\frac{A}{2} - \tan\frac{B}{2}\right)$$

33. Show that $\cos 5\theta - \sin 5\theta$

$$= 16\cos(\theta - 27^\circ)\cos(\theta + 9^\circ)\sin(\theta + 27^\circ)\sin(\theta - 9^\circ)(\cos\theta - \sin\theta).$$

34. Resolve the following expression into factors; and determine what relation must exist among the angles in order that the expression may vanish:—

$$\cos^2 A + \cos^2 B + \cos^2 C - 2\cos A.\cos B.\cos C - 1$$

35. Prove the following formulæ :—

$$\sec\left(\frac{\pi}{4} + x\right)\sec\left(\frac{\pi}{4} - x\right) = 2\sec 2x$$

$$\sin\left(\frac{\pi}{3} + x\right) - \sin\left(\frac{\pi}{3} - x\right) = \sin x$$

36. Show how to find x from the equation,

$$\frac{m \,.\, \tan(a - x)}{n \tan x} = \left\{\frac{\cos x}{\cos(a - x}\right\}^2$$

37. Show that

$$\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A \,.\, \cos B \,.\, \cos C - 1 =$$
$$4\cos\tfrac{1}{2}(A+B+C).\cos\tfrac{1}{2}(B+C-A).\cos\tfrac{1}{2}(A+C-B).\cos\tfrac{1}{2}(A+B-C)$$

38. Given $\sin\theta + \sin\phi = a$
$\cos\theta + \cos\phi = b$

find the values of $\cos(\theta + \phi)$ and $\cos(\theta - \phi)$ in terms of a and b.

39. Show that $\dfrac{\sin 2A}{1 + 2\cos A + \cos 2A} = \tan\tfrac{1}{2}A.$

40. Eliminate θ from

$$x\cos\theta + y\sin\theta = p = x\cos 2\theta + y\sin 2\theta.$$

41. Show that, unless $A = \dfrac{(2K + 1)\pi}{r}$

$$\frac{\sin(n - r)A + 2\sin nA + \sin(n + r)A}{\sin(m - r)A + 2\sin mA + \sin(m + r)A} = \frac{\sin nA}{\sin mA}$$

and explain the reason for the exception.

42. If $A + B + C = (3 \pm 1)\dfrac{\pi}{2}$, then

$$1 \pm 2\cos A \,.\, \cos B \,.\, \cos C = \cos^2 A + \cos^2 B + \cos^2 C.$$

43. Prove that

$$\frac{\sin a + 2\sin 3a + \sin 5a}{\sin 3a + 2\sin 5a + \sin 7a} = \frac{\sin 3a}{\sin 5a}$$

44. If $\left.\begin{aligned} &\frac{x^2 - y^2}{a^2}\cos\theta = \frac{z^2}{b^2}\cos\phi \\ &\frac{x}{\sin(\theta + \phi)} = \frac{y}{\sin(\theta - \phi)} = \frac{z}{\sin 2\theta} \end{aligned}\right\}$

then shall $\dfrac{\sin\theta}{\sin\phi} = \dfrac{b^2}{a^2}$

45. If $\phi = \dfrac{\pi}{13}$, prove that

$$\cos 5\phi + \cos 7\phi + \cos 11\phi = \frac{1 - \sqrt{13}}{4}$$

46. Prove geometrically that
$\sin(A+B) : \sin(A-B) :: \tan A + \tan B : \tan A - \tan B.$

47. Prove the following formulæ:—

(a) $\tan\alpha \cdot \tan(\alpha-\beta) = \dfrac{\tan\alpha - \tan\beta}{\cot\alpha + \tan\beta}$

(b) $\dfrac{\sin\alpha + \sin\beta}{\cos\alpha + \cos\beta} = \dfrac{\operatorname{vers}(\alpha+\beta)}{\sin(\alpha+\beta)}$

(c) $chd\,(\pi - \theta) = \dfrac{chd\,2\theta}{chd\,\theta}$

(d) $2(\sin^2\alpha\sin^2\beta + \cos^2\alpha\cos^2\beta) = 1 + \cos 2\alpha \cdot \cos 2\beta.$

48. If $A + B + C = 180°$, then

$$\cos A + \cos B + \cos C = 4\sin\frac{A}{2} \cdot \sin\frac{B}{2} \cdot \sin\frac{C}{2} + 1.$$

49. Eliminate α and β from the equations,

$$\left.\begin{aligned} a\sin^2\alpha + b\cos^2\alpha &= m \\ a\cos^2\beta + b\sin^2\beta &= n \\ a\tan\alpha &= b\tan\beta \end{aligned}\right\}$$

50. If $\cos(A - C) \cdot \cos B = \cos(A + C - B)$
then $\tan A$, $\tan B$, $\tan C$ are in harmonical progression.

51. Show that $\sin(30° + B) = \cos B - \sin(30° - B)$.

52. Prove the following formulæ:—

(a) $\sin A \cdot \tan\dfrac{A}{2} = \operatorname{versin} A$

(b) $\dfrac{\sin A + \sin 3A}{\cos A + \cos 3A} = \tan 2A.$

53. Solve the following equations:—

(a) $a\sin\alpha + b\cos\alpha = c$

(b) $\left.\begin{aligned} \sin\alpha + \sin\beta &= m \\ \cos\alpha + \cos\beta &= n \end{aligned}\right\}$

54. Solve the following equations:—

(a) $\sin\alpha + \sin 2\alpha + \sin 3\alpha = 0$

(b) $\cot\left(\dfrac{\pi}{4} - \alpha\right) = 3\cot\left(\dfrac{\pi}{4} + \alpha\right)$

55. Prove the following:—

(a) $\dfrac{2(\sin\alpha + \cos\alpha)}{\cos\dfrac{\alpha}{2} - \sin\dfrac{\alpha}{2} - \cos\dfrac{3\alpha}{2} + \sin\dfrac{3\alpha}{2}} = \operatorname{cosec}\dfrac{\alpha}{2}$

(b) $\dfrac{\sin\alpha + \sin n\alpha + \sin(2n-1)\alpha}{\cos\alpha + \cos n\alpha + \cos(2n-1)\alpha} = \tan n\alpha.$

56. Prove the following :—

$$(a)\ \sin 4\alpha = 2 \sin 3\alpha \cdot \cos \alpha - \sin 2\alpha$$
$$(b)\ \sin 5\alpha = 2 \sin 4\alpha \cdot \cos \alpha - \sin 3\alpha.$$

57. Prove the following :—If $\alpha + \beta + \gamma = 180°$

(a) $\cos 4\alpha + \cos 4\beta + \cos 4\gamma = 4 \cos 2\alpha \cdot \cos 2\beta \cdot \cos 2\gamma - 1$

(b) $\sin \frac{\alpha}{2} + \sin \frac{\beta}{2} + \sin \frac{\gamma}{2} = 4 \sin \frac{\pi - \alpha}{2} \cdot \sin \frac{\pi - \beta}{2} \cdot \sin \frac{\pi - \gamma}{2} + 1$

(c) $\dfrac{\sin \alpha + \sin \beta - \sin \gamma}{\sin \alpha + \sin \beta + \sin \gamma} = \tan \frac{\alpha}{2} \cdot \tan \frac{\beta}{2}$

58. Show that

$$\sin n\alpha - \sin (n - 2)\alpha = 2 \cos (n - 1)\alpha \cdot \sin \alpha$$
$$\cos (n - 2)\alpha - \cos n\alpha = 2 \sin (n - 1)\alpha \cdot \sin \alpha.$$

59. What values of α will satisfy the equation,

$$\sin \beta + \sin (2\alpha + \beta) - \sin (2\alpha - \beta) = \sin (\alpha + \beta) - \sin (\alpha - \beta)$$

60. Find x from the equation,

$$x \tan \alpha = (\sqrt{1 + x} - 1)(\sqrt{1 - x} + 1).$$

61. Show that $\dfrac{\sec (2^{2n+1}\alpha) - 1}{\sec (2^{2n}\alpha) - 1} = \dfrac{\cot 2^{2n-1}\alpha}{\cot (2^{2n+1}\alpha)}$

62. Show that

$$\left(\frac{\tan \alpha}{\tan \beta} - \frac{\tan \beta}{\tan \alpha}\right) + \left(\frac{\tan \beta}{\tan \gamma} - \frac{\tan \gamma}{\tan \beta}\right) + \left(\frac{\tan \gamma}{\tan \alpha} - \frac{\tan \alpha}{\tan \gamma}\right)$$
$$= \frac{8 \sin (\alpha - \beta) \cdot \sin (\beta - \gamma) \cdot \sin (\gamma - \alpha)}{\sin 2\alpha \cdot \sin 2\beta \cdot \sin 2\gamma}$$

63. Show that

$$\cos^2 \alpha + \cos^2 \left(\frac{\pi}{3} - \alpha\right) + \cos^2 \left(\frac{\pi}{3} + \alpha\right) = 1\tfrac{1}{2}.$$

64. If $\dfrac{a \tan \alpha + b \tan \beta}{(a + b)} = \tan \tfrac{1}{2} (\alpha + \beta)$

$$\text{then } \frac{a}{b} = \frac{\cos \alpha}{\cos \beta}.$$

65. If $\tan A = \dfrac{1}{\sqrt{3}} \cdot \tan B = \dfrac{1}{\sqrt{15}}$

then $\sin (A + B) = \sin 60° \cos 36°$.

66. If $\cos A = \cos B \cdot \cos C$. then

$$\tan \frac{A + B}{2} \cdot \tan \frac{A - B}{2} \cdot = \tan^2 \frac{C}{2}$$

67. If $A + B + C = \frac{\pi}{2}$ and A, B, C be in arithmetical progression, then

$$\sqrt{3} - \tan A = (1 + \sqrt{3} \tan A) \tan C.$$

68. If $A + B + C = \frac{\pi}{2}$. then

$$\tan A + \tan B + \tan C = \tan A . \tan B . \tan C + \sec A . \sec B . \sec C.$$

69. Show that

$$chd\ n A . chd\ (\pi - A) = chd\ (n + 1) A + chd\ (n - 1) A.$$

70. Show that

$$\sin^2 (A + B) - \sin^2 (A - B) = \sin 2 A . \sin 2 B.$$

CHAPTER VI.

ON THE DIVISION OF ANGLES.

1. Since

$$2 \sin \frac{\alpha}{2} . \cos \frac{\alpha}{2} = \sin \alpha$$

$$\sin^2 \frac{\alpha}{2} + \cos^2 \frac{\alpha}{2} = 1$$

hence, by addition and subtraction,

$$\left(\sin \frac{\alpha}{2} + \cos \frac{\alpha}{2}\right)^2 = 1 + \sin \alpha$$

$$\left(\sin \frac{\alpha}{2} - \cos \frac{\alpha}{2}\right)^2 = 1 - \sin \alpha$$

$$\text{or, } \sin \frac{\alpha}{2} + \cos \frac{\alpha}{2} = \pm \sqrt{1 + \sin \alpha} \quad \ldots\ldots 1$$

$$\sin \frac{\alpha}{2} - \cos \frac{\alpha}{2} = \pm \sqrt{1 - \sin \alpha} \quad \ldots\ldots 2$$

$$\therefore\ 2 \sin \frac{\alpha}{2} = \pm \sqrt{1 + \sin \alpha} \pm \sqrt{1 - \sin \alpha}$$

$$2 \cos \frac{\alpha}{2} = \pm \sqrt{1 + \sin \alpha} \mp \sqrt{1 - \sin \alpha}$$

Hence, we see that for one value of $\sin a$ we have four values for $\sin \frac{a}{2}$ and $\cos \frac{a}{2}$.

Now suppose the angle a to lie between 180° and 270° $\therefore \frac{a}{2}$ lies between 90° and 135°; and in this case we know that $\sin \frac{a}{2}$ is *positive* and $\cos \frac{a}{2}$ *negative*, and $\sin \frac{a}{2}$ is greater than $\cos \frac{a}{2}$.

$$\therefore \sin \frac{a}{2} + \cos \frac{a}{2} = + \sqrt{1 + \sin a}$$

$$\sin \frac{a}{2} - \cos \frac{a}{2} = + \sqrt{1 - \sin a}$$

$$\therefore 2 \sin \frac{a}{2} = \sqrt{1 + \sin a} + \sqrt{1 - \sin a}$$

$$2 \cos \frac{a}{2} = \sqrt{1 + \sin a} - \sqrt{1 - \sin a}.$$

2. Again, suppose the angle a to lie between 90° and 180°, then $\frac{a}{2}$ lies between 45° and 90°. In this case $\sin \frac{a}{2}$ and $\cos \frac{a}{2}$ are both positive, being in the 1st quadrant, and the sine is greater then the cosine, hence

$\sin \frac{a}{2} + \cos \frac{a}{2}$ is positive; so also is $\sin \frac{a}{2} - \cos \frac{a}{2}$

$$\therefore \sin \frac{a}{2} + \cos \frac{a}{2} = + \sqrt{1 + \sin a}$$

$$\sin \frac{a}{2} - \cos \frac{a}{2} = + \sqrt{1 - \sin a}$$

$$\therefore 2 \sin \frac{a}{2} = \sqrt{1 + \sin a} + \sqrt{1 - \sin a}$$

$$2\cos\frac{a}{2} = \sqrt{1+\sin a} - \sqrt{1-\sin a}$$

3. If a lay between 270° and 360°, then $\frac{a}{2}$ lies between 135° and 180°. Here $\sin\frac{a}{2}$ is positive and $\cos\frac{a}{2}$ negative, and the sine is less than the cosine.

$$\therefore \sin\frac{a}{2} + \cos\frac{a}{2} \text{ is negative}$$

$$\sin\frac{a}{2} - \cos\frac{a}{2} \text{ is positive.}$$

Hence,
$$\sin\frac{a}{2} + \cos\frac{a}{2} = -\sqrt{1+\sin a}$$

$$\sin\frac{a}{2} - \cos\frac{a}{2} = +\sqrt{1-\sin a}$$

$$\therefore 2\sin\frac{a}{2} = -\sqrt{1+\sin a} + \sqrt{1-\sin a}$$

$$2\cos\frac{a}{2} = -\sqrt{1+\sin a} - \sqrt{1-\sin a}$$

4. The student will see the application of these formulæ by a few practical questions. Given the sine of 45°, to find the trigonometrical functions of $22\frac{1}{2}°$.

Now this angle $22\frac{1}{2}°$ is in the 1st quadrant. Both sin and cos are positive, and the cos is the greater.

$\therefore \sin\frac{a}{2} + \cos\frac{a}{2}$ and $\sin\frac{a}{2} - \cos\frac{a}{2}$ are positive and negative respectively.

$$\sin 22\tfrac{1}{2}° + \cos 22\tfrac{1}{2}° = +\sqrt{1+\sin 45°}$$
$$\sin 22\tfrac{1}{2}° - \cos 22\tfrac{1}{2}° = -\sqrt{1-\sin 45°}$$

$$\therefore 2\sin 22\tfrac{1}{2}° = \sqrt{1+\frac{1}{\sqrt{2}}} - \sqrt{1-\frac{1}{\sqrt{2}}}$$

$$2\cos 22\tfrac{1}{2}° = \sqrt{1+\frac{1}{\sqrt{2}}} + \sqrt{1-\frac{1}{\sqrt{2}}}$$

In order to get a simpler expression for each of these, square each,

$$\therefore\ 4 \sin^2 22\tfrac{1}{2}^\circ = 2 - 2\sqrt{\tfrac{1}{2}} = 2 - \sqrt{2}$$

$$\therefore\ \sin 22\tfrac{1}{2}^\circ = \frac{\sqrt{2 - \sqrt{2}}}{2}$$

$$\text{and } \cos 22\tfrac{1}{2}^\circ = \frac{\sqrt{2 + \sqrt{2}}}{2}$$

5. Find the sin and cos 165° from the known values of sin 330°.

Now this angle lies between 135° and 225°, and from the figure it can be seen that sin 165° is positive, whereas cos 165° is negative, and numerically greater than the sin 165°.

$$\therefore\ \sin 165^\circ + \cos 165^\circ = -\sqrt{1 + \sin 330^\circ}$$

$$\sin 165^\circ - \cos 165^\circ = \sqrt{1 - \sin 330}$$

$$\therefore\ 2 \sin 165^\circ = -\sqrt{1 + \sin 330} + \sqrt{1 - \sin 330^\circ}$$

$$\text{now } \sin 330 = -\sin(360 - 330) = -\sin 30$$

$$\therefore\ \sin 330 = -\tfrac{1}{2}$$

$$\therefore\ 2 \sin 165^\circ = -\sqrt{1 - \tfrac{1}{2}} + \sqrt{1 + \tfrac{1}{2}}$$

$$= -\frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{\sqrt{2}} = \frac{\sqrt{3} - 1}{\sqrt{2}}$$

$$\therefore\ \sin 165^\circ = \frac{\sqrt{3} - 1}{2\sqrt{2}}$$

$$\text{and } \cos 165^\circ = -\frac{\sqrt{3} + 1}{2\sqrt{2}}$$

6. Find tan 165° from the known value of tan 330°.

$$\text{Tan } 2A = \frac{2 \tan A}{1 - \tan^2 A}\ \therefore\ \tan 330^\circ = \frac{2 \tan 165^\circ}{1 - \tan^2 165^\circ}$$

$$\text{but } \tan 330 = -\tan(360 - 330) = -\tan 30 = -\frac{1}{\sqrt{3}}$$

$$\therefore\ \frac{2 \tan 165^\circ}{1 - \tan^2 165^\circ} = -\frac{1}{\sqrt{3}}, \text{ from this we get}$$

$$\tan 165^\circ = \sqrt{3} - 2.$$

7. Some questions similar to the following may be put:—

Between what limits must $\frac{a}{2}$ lie so that

$$2 \sin \frac{a}{2} = -\sqrt{1 + \sin a} + \sqrt{1 - \sin a}\,?$$

$$\text{Here } \sin \frac{a}{2} + \cos \frac{a}{2} = -\sqrt{1 + \sin a}$$

$$\sin \frac{a}{2} - \cos \frac{a}{2} = +\sqrt{1 - \sin a}$$

therefore $\cos \frac{a}{2}$ should be negative, and greater than $\sin \frac{a}{2}$,

and this is the case when $\frac{a}{2}$ lies between 135° and 225°.

8. Give the arithmetical value and algebraical sign of $\sin a \,.\, \cos a$ for every multiple of 30° up to 360°.

In 1st quadrant $\sin a$ is positive and $\cos a$ positive

if $a = 30° \therefore \sin a \,.\, \cos a$ is + and $= \frac{1}{2} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4}$

if $a = 60° \therefore \sin a \,.\, \cos a$ is + and $= \frac{\sqrt{3}}{2} \cdot \frac{1}{2} = \frac{\sqrt{3}}{4}$

if $a = 90° \therefore \sin a \,.\, \cos a$ is + and $= 1 \,.\, 0 = 0$

In 2nd quadrant $\sin a$ is positive and $\cos a$ negative

if $a = 120° \therefore \sin a \cos a$ is − and $= \frac{\sqrt{3}}{2} \,.\, \times -\frac{1}{2}$

$$= -\frac{\sqrt{3}}{4}$$

if $a = 150° \therefore \sin a \,.\, \cos a$ is − and $= \frac{1}{2} \times -\frac{\sqrt{3}}{2}$

$$= -\frac{\sqrt{3}}{4}$$

if $a = 180° \therefore \sin a \,.\, \cos a$ is − and $= 0 \times -1 = 0$

EXERCISES.

1. Find the sin, cos, and tan of $22\frac{1}{2}^\circ$.
2. Find the sin, cos, and tan of $7\frac{1}{2}^\circ$.
3. Find the tan $142\frac{1}{2}^\circ$.
4. If $\cos a + \cos 2a + \cos 3a = 0$, find a.
5. Trace cos A in sign and magnitude from 0° to 360°. Find an expression for all angles having the same cosine.
6. Find the sin and cos of 9°.
7. Find the sin and cos of 3°.
8. Find expressions for the *sin* and *cos* of the following angles:—
   3°, 9°, 15°, 18°, 72°, 36°, 54°, 75°, 45°, 30°, 60°, 81°.
9. Given $\cot \frac{a}{2} = 2 + \sqrt{3}$, find $\cos a$.
10. Between what limits must A lie in order to satisfy the following equations:—

$$\sin \frac{A}{2} = \tfrac{1}{2} \left\{ \sqrt{1 - \sin A} - \sqrt{1 + \sin A} \right\}$$

$$\cos \frac{A}{2} = - \tfrac{1}{2} \left\{ \sqrt{1 + \sin A} + \sqrt{1 - \sin A} \right\}$$

11. Determine the limits between which A must lie to satisfy the equations—

$$2 \sin \frac{A}{2} = \sqrt{1 - \sin A} - \sqrt{1 + \sin A}$$

$$2 \cos \frac{A}{2} = - \sqrt{1 - \sin A} - \sqrt{1 + \sin A}$$

12. Show that

$$2 \cos \frac{A}{2} = (-1)^m \sqrt{1 + \sin A} + (-1)^n \sqrt{1 - \sin A}$$

where m and n are the greatest integers contained in

$$\frac{A^\circ + 90^\circ}{360^\circ} \text{ and } \frac{A^\circ + 270^\circ}{360^\circ} \text{ respectively.}$$

13. Find the *sin* and *cos* 165° from the known value of sin 330°.
14. Show that

$$2 \cos \frac{A}{2} = \sqrt{1 + \sin A} - \sqrt{1 - \sin A}$$

What is the use of this formula? How many values of $\cos \frac{A}{2}$ does it give? If A lies between 135° and 180°, how must the signs be taken?

15. Prove that

$$\cos\frac{a}{2} = -\sqrt{\frac{1+\cos a}{2}}$$

$$\sin\frac{a}{2} = -\sqrt{\frac{1-\cos a}{2}}$$

when a lies between 360° and 540°; given

$$2\cos^2\frac{a}{2} = 1 + \cos a$$

$$2\sin^2\frac{a}{2} = 1 - \cos a$$

16. Between what limits must A lie in order that

$$2\sin\frac{A}{2} = -\sqrt{1+\sin A} - \sqrt{1-\sin A}?$$

17. Find the value of cos 11° 15′.

CHAPTER VII.

LOGARITHMS.

LOGARITHMIC SERIES.

1. Suppose we take a series of numbers in geometrical progression, as

$$2^0,\ 2^1,\ 2^2,\ 2^3,\ 2^4,\ 2^5,\ 2^6,\ \&c.,$$

we see that the indices are in arithmetical progression.

Indices are .	0	1	2	3	4	5	6
Numbers .	1	2	4	8	16	32	64

Multiplication and division of the second line are performed by adding or subtracting their respective indices in the upper line; thus, 4×16 being found under $2+4$, or 6 is 64; $32 \div 4$ being found under $5 - 2$, or 3 is 8; and so on.

The square root of 64 is found under $6 \div 2$ or 3, and is consequently 8.

The cube root of 64 is found under $6 \div 3$ or 2, and is therefore 4.

The numbers in the upper line are called the logarithms of those in the lower line, 2 being the base.

$$\text{Thus } \log_2 4 = 2$$
$$\log_2 64 = 6$$
$$\log_2 2 = 1, \text{ and so on.}$$

Hence the definition. *The logarithm of any number to any base is the power to which this base must be raised in order to give the number.*

The base we generally use is 10.

$$\text{And } 10^0 = 1 \qquad \therefore \log_{10} 1 = 0$$
$$10^1 = 10 \qquad \therefore \log_{10} 10 = 1$$
$$10^2 = 100 \qquad \therefore \log_{10} 100 = 2$$

so that log 1 to any base is always 0, and log any number to same base is 1.

2. *The logarithm of the product of two numbers is equal to the sum of their logarithms.*

$$\text{Thus, } \log (4 \times 9) = \log 4 + \log 9.$$

To prove this generally, let M and N be the two numbers and a the base; then

$$\text{let } a^x = \text{M} \quad \ldots\ldots\ldots\ldots \quad (1)$$
$$a^y = \text{N} \quad \ldots\ldots\ldots\ldots \quad (2)$$
$$\therefore x = \log_a \text{M}$$
$$y = \log_a \text{N}$$
$$\therefore x + y = \log_a \text{M} + \log_a \text{N}$$

and by multiplying (1) and (2) we get

$$a^{x+y} = \text{MN}$$
$$\therefore x + y = \log_a \text{M N}$$
$$\text{Hence } \log_a \text{MN} = \log_a \text{M} + \log_a \text{N}.$$

3. *The logarithm of the quotient of two numbers is equal to the logarithm of the dividend minus the logarithm of the divisor.*

$$\text{Thus, } \log_{10}\left(\frac{17}{4}\right) = \log_{10} 17 - \log_{10} 4$$

To prove this generally, as before,

$$\text{let } a^x = \text{M} \quad \ldots\ldots\ldots\ldots (1)$$
$$a^y = \text{N} \quad \ldots\ldots\ldots\ldots (2)$$
$$\therefore\ x = \log_a \text{M}$$
$$y = \log_a \text{N}$$
$$\therefore\ x - y = \log_a \text{M} - \log_a \text{N}$$

and by dividing (1) by (2) we get $a^{x-y} = \dfrac{\text{M}}{\text{N}}$

$$\therefore\ x - y = \log_a \left(\frac{\text{M}}{\text{N}}\right)$$

$$\text{Hence } \log_a \left(\frac{\text{M}}{\text{N}}\right) = \log_a \text{M} - \log_a \text{N}.$$

4. *The logarithm of a number raised to any power is equal to that power multiplied by the logarithm of the number.*

$$\text{Thus, } \log_{10} 12^5 = 5 \log_{10} 12.$$

To prove this generally,

$$\text{let } a^x = \text{M} \quad \ldots\ldots\ldots\ldots (1)$$
$$\therefore\ x = \log_a \text{M}$$
$$\therefore\ px = p \log_a \text{M}$$

and by raising both sides of (1) to the p^{th} power we get

$$a^{px} = \text{M}^p$$
$$\therefore\ px = \log_a \text{M}^p$$

$$\text{Hence } \log_a \text{M}^p = p \log_a \text{M}.$$

5. *To find the relation that exists between the logarithm of the same number M to two different bases, a and b,*

$$\text{let } a^x = \text{M}$$
$$b^y = \text{M}$$
$$\therefore\ x = \log_a \text{M}$$
$$y = \log_b \text{M}$$

$$\text{Also } a^x = b^y$$
$$\therefore a^{\frac{x}{y}} = b$$
$$b^{\frac{y}{x}} = a$$

$$\therefore \frac{x}{y} = \log_a b$$

$$\frac{y}{x} = \log_b a \qquad \therefore \log_b a \times \log_a b = 1$$

$$\therefore \log_b a = \frac{1}{\log_a b} \text{ or } \log_b 10 = \frac{1}{\log_{10} b}$$

$$\text{Also } x = y \,.\, \log_a b$$
$$\text{or } \log_a M = \log_b M \times \log_a b$$
$$\text{or } \log_{10} M = \log_b M \times \log_{10} b$$
$$\text{or } \log_{10} M = \log_b M \times \frac{1}{\log_b 10.}$$

Hence we see that the logarithms of any number to two different bases 10 and ϵ are connected by a constant multiplier or modulus $\left(\frac{1}{\log_\epsilon 10}\right)$, so that we can find the logarithms of numbers to the base 10 from those already known to the base ϵ.

6. In Briggs', or the common system of logarithms, if the logarithm of any number be given, we can determine the logarithm of the product of that number by 10, or any power of 10, or the logarithm of the quotient of that number by any power of 10.

$$\text{Thus, } \log_{10}(10^n \times N) = \log_{10} 10^n + \log_{10} N$$
$$= n + \log_{10} N$$
$$\text{and } \log_{10}\left(\frac{N}{10^n}\right) = \log_{10} N - \log_n 10^n$$
$$= \log_{10} N - n.$$

The integral part of the logarithm is called the *characteristic*, and the decimal part the *mantissa*.

To illustrate what we have just stated, let the logarithm of any number, say 3, be known.

$$\log 3 = \cdot 4771213$$

then

$$\log 3 \times 10 = \log 30 = \log 3 + 1 = 1\cdot 4771213$$
$$\log 3 \times 10^2 = \log 300 = \log 3 + 2 = 2\cdot 4771213$$
$$\log 3 \times 10^5 = \log 300000 = \log 3 + 5 = 5\cdot 4771213$$
$$\log \frac{3}{10} = \log \cdot 3 = \log 3 - 1 = \bar{1}\cdot 4771213$$
$$\log \frac{3}{1000} = \log \cdot 003 = \log 3 - 3 = \bar{3}\cdot 4771213$$

and so on.

The negative sign in such cases as the last two is always written *over* the *characteristic.*

7. To determine by inspection the *characteristic* of the logarithm of any given number.

Suppose the number lies between

1 and 10,	its log lies between 0 and 1	∴	characteristic is 0
10 and 100	,, 1 and 2	∴	,, 1
1000 and 10000	,, 3 and 4	∴	,, 3

Therefore we see that generally, if the number lies between 10^n and 10^{n+1}, its characteristic must lie between n and $(n + 1)$, and is consequently (n); that is, in such cases the characteristic is less by *one* than the number of integral figures in the number.

Again,

If the No. be between	its log is between	and its characteristic
1 and $\frac{1}{10}$ or $\cdot 1$	0 and -1 . . .	-1
$\frac{1}{10}$ and $\frac{1}{100}$ or $\cdot 01$	-1 ,, -2 . . .	-2
$\frac{1}{1000}$ and $\frac{1}{10000}$ or . 0001	-3 ,, -4 . . .	-4
$\frac{1}{10^n}$ and $\frac{1}{10^{n+1}}$	$-n$,, $-(n+1)$	$-(n+1)$

So that in cases of this kind the *characteristic* is always *negative*, and numerically *one more* than the number of ciphers which come immediately after the decimal point.

8. The advantage of this system can be seen at once; for the mantissa of $\log N \times 10^n$ and $\log N \div 10^n$ are the same, whereas if any other base were taken the mantissa of the one would not be that of the other.

9. *To expand a^x in a series ascending by powers of x, or, in other words, to expand the number in a series ascending by powers of the logarithm.*

$$\text{Now } a^x = \{1 + (a-1)\}^x$$
$$= 1 + x(a-1) + x.\frac{x-1}{2}(a-1)^2$$
$$+ \frac{x.(x-1)(x-2)}{1.2.3}(a-1)^3 + \ldots\ldots$$

Let the co-efficient of x be denoted by p, and the co-efficients of x^2, x^3, by p', p'', then

$$a^x = 1 + px + p'x^2 + p''x^3 + \ldots\ldots$$
$$\text{also } a^z = 1 + pz + p'z^2 + p''z^3 + \ldots\ldots$$
$$\text{similarly } a^{x+z} = 1 + p(x+z) + p'(x+z)^2 + p''(x+z)^3 + \ldots\ldots\ldots\ldots$$
$$\text{and since } a^{x+z} = a^x \times a^z$$
$$\therefore 1 + p(x+z) + p'(x+z)^2 + p''(x+z)^3 + \ldots$$
$$= \{1 + px + p'x^2 + p''x^3 + \ldots\ldots\}$$
$$\times \{1 + pz + p'z^2 + p''z^3 + \ldots\ldots\}$$

and equating the co-efficients of the terms involving xz, x^2z, x^3z . . . . $x^{n-1}z$;

$$\text{thus } 2p' = p^2 \quad \therefore p' = \frac{p^2}{2}$$
$$3p'' = pp' \therefore p'' = \frac{p^3}{1.2.3}$$
$$4p''' = pp'' \therefore p''' = \frac{p^4}{1.2.3.4}$$

. . . .

Then $a^x = 1 + p\,x + \frac{p^2\,x^2}{1.2} + \frac{p^3\,x^3}{1.2.3} + \ldots\ldots$

Now if $p\,x = 1 \;\therefore\; x = \frac{1}{p}$

$\therefore\; a^{\frac{1}{p}} = 1 + 1 + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \ldots$

This series is generally denoted by ϵ, so that

$\epsilon = 1 + 1 + \frac{1}{1.2} + \frac{1}{\underline{|3}} + \frac{1}{\underline{|4}} + \frac{1}{\underline{|5}} + \frac{1}{\underline{|6}} + \ldots$

or $\epsilon = 2\cdot7182818$.

Again, since $a^{\frac{1}{p}} = \epsilon \;\therefore\; a = \epsilon^p \;\therefore\; p = \log_\epsilon a$, so that

$a^x = 1 + (\log_\epsilon a)\,x + \frac{(\log_\epsilon a)^2\,x^2}{\underline{|2}} + \frac{(\log_\epsilon a)^3\,x^3}{\underline{|3}} + \ldots$

If the base be ϵ, this last becomes

$\epsilon^x = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \frac{x^4}{\underline{|4}} + \ldots\ldots\ldots\ldots$

10. To expand $\log_\epsilon (1 + x)$ in a series of ascending powers of x.

From the last article, since

p = the co-efficient of x

$= a - 1 - \frac{(a-1)^2}{2} + \frac{(a-1)^3}{3} - \frac{(a-1)^4}{4} + \ldots\ldots$

also $p = \log_\epsilon a$

and instead of a write $(1 + x)$

$\therefore\; \log_\epsilon (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\ldots$

11. To find $\log_\epsilon 10$ and the modulus $\frac{1}{\log_\epsilon 10}$.

From last article,

$\log_\epsilon (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\ldots$

also $\log_\epsilon (1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} \ldots\ldots\ldots$

By subtraction,

$$\log_\epsilon\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \ldots\ldots\right)$$

If for x we write $\dfrac{a-b}{a+b}$

consequently we must write $\dfrac{a}{b}$ for $\dfrac{1+x}{1-x}$

then $\log_\epsilon \dfrac{a}{b} = 2\left\{\dfrac{a-b}{a+b} + \dfrac{1}{3}\left(\dfrac{a-b}{a+b}\right)^3 + \dfrac{1}{5}\left(\dfrac{a-b}{a+b}\right)^5 + \ldots\right\}$

If $b = 1$

$$\therefore \log_\epsilon a = 2\left\{\frac{a-1}{a+1} + \frac{1}{3}\left(\frac{a-1}{a+1}\right)^3 + \frac{1}{5}\left(\frac{a-1}{a+1}\right)^5 + \ldots\right\}$$

From this we can calculate the log of any number to the base ϵ. If $a = 3$

$$\therefore \log_\epsilon 3 = 1\cdot 0986122$$

$$\text{and } \log_\epsilon 10 = 2\cdot 3025850$$

$$\text{and } \therefore \frac{1}{\log_\epsilon 10} = \cdot 4342944.$$

Again, if $a = b + 1$

$$\therefore \log_\epsilon \frac{b+1}{b} \text{ or } \log_\epsilon (b+1) - \log_\epsilon b$$

$$= 2\left\{\frac{1}{2b+1} + \frac{1}{3(2b+1)^3} + \frac{1}{5(2b+1)^5} + \ldots\right\}$$

From this we can calculate the logarithm of any of two consecutive numbers when the logarithm of the other is given. We could also calculate $\log_\epsilon 10$ from this from the known $\log_\epsilon 9$ or $2 \log_\epsilon 3$.

All logarithms to this base ϵ are called *Naperian* logarithms, after their inventor, Napier.

USE OF LOGARITHMIC TABLES.

12. The tables of logarithms are generally calculated up to seven places of decimals. The following may be taken

as a general specimen of the way in which the logarithms of numbers are tabulated :—

No.	0	1	2	3	4	5	6	7	8	9	D
2610	4166405	6571	6738	6904	7071	7237	7403	7570	7736	7902	167
11	8069	8235	8401	8568	8734	8900	9067	9233	9399	9565	
12	9732	9898	$\bar{0}064$	0231	0397	0563	0729	0895	1062	1228	
13	4171394	1560	1726	1893	2059	2225	2391	2557	2724	2890	
14	· 3056	3222	3388	3554	3720	3886	4053	4219	4385	4551	
Diff. } 167	Prpts. }	17	33	50	67	84	100	117	134	150	

So that if we have to find the logarithm of a number consisting of four figures, we have only to look in the column headed by 0, prefix a decimal point and the proper characteristic.

$$\begin{aligned} \text{Thus, } \log\ 26120 &= 4 \cdot 4169732 \\ \log 2 \cdot 6120 &= \ \cdot 4169732 \\ \log\ \cdot 0026120 &= \bar{3} \cdot 4169732 \end{aligned}$$

If the number consists of five figures, we can look for the four first figures in the column headed No., and the fifth in the top column, and opposite the four first figures, and under the fifth, will be found the logarithm.

Thus, find the logarithm of 26·136

$$\log 26{\cdot}136 = 1 \cdot 4172391$$

If the number consists of six figures, find the logarithm of the first five as before, and multiply the sixth figure by the tabular difference in the column marked D, and reject the last figure; but if this last figure be *greater* than 5, then the preceding number must be increased by 1.

Find log 261 · 4 57

$$\begin{aligned} \log 261 \cdot 45 &= 2 \cdot 4173886 \\ \text{Tab. diff.} &= 167 \\ \text{and this multiplied by } 7 &= \underline{\qquad\ 117} \\ \therefore \log 261 \cdot 457 &= 2 \cdot 4174003 \end{aligned}$$

When the first three figures of a logarithm have once

been inserted, they must be understood as repeated before every four figures in the other columns until they are superseded by a higher number. When this change has to be made at any place other than the commencement of one of the horizontal rows, then the first of the four figures corresponding to this change is written with a stroke over it: thus, in column 2, we have $\bar{0}064$, showing that hereafter we must not prefix 416, but 417.

13. *To find the logarithm of a number which is not contained in the table.*

Suppose the logarithm of 2611234 is required.

Here from the tables we can take

$$\begin{aligned} \log 2611300 &= 6 \cdot 4168568 \\ \log 2611200 &= 6 \cdot 4168401 \\ \text{Difference for } 100 &= \quad \cdot 0000167 \end{aligned}$$

Now for an increase in the numbers we have a proportional increase in the logarithm; hence, as

$$100 : 34 :: \cdot 0000167 : x$$

$$\therefore x = \frac{34}{100} \times \cdot 0000167 = \cdot 0000057$$

$$\therefore \log 2611234 = 6 \cdot 4168401 + \cdot 0000057 = 6 \cdot 4168458$$

This, we see, corresponds with the rule before given in the last article.

14. *To find the number corresponding to a given logarithm.*

When the logarithm is found exactly in the tables the number is found in the left column headed No. and opposite to the given logarithm, and the fifth figure of the number at the top of the column in which it is found.

But if the logarithm be not exactly found in the tables, we take the next lower logarithm and the number consisting of five figures which belongs to it; we then find the difference between the two logarithms, annex ciphers, and divide by the tabular difference, and we thus get the sixth, seventh, &c., figures of the number.

Thus, required the number of which the logarithm is $3 \cdot 9232670$.

$$\begin{aligned} \text{Given log} &= \cdot 9232670 \\ \text{next lower} &= \underline{\cdot 9232647} = \log 83804 \\ &\qquad 23 \end{aligned}$$

$$\text{tabular difference} = 52 \text{ and } \frac{230000}{52} = 44, \text{ \&c.}$$

$$\therefore \text{ the required number} = 8380 \cdot 444$$

The reason of this is,

$$\begin{aligned} \text{next higher log} &= \cdot 9232699 \\ \text{next lower log} &= \cdot 9232647 \end{aligned}$$

$$\begin{aligned} \text{Number corresponding to first log} &= 83805 \\ \text{,, \quad ,, \quad second log} &= 83804 \\ \text{Difference in two logs} &= \cdot 0000052 \end{aligned}$$

and excess of given log over $\cdot 9232647 = \cdot 0000023$, and the difference in the numbers being proportional to the difference in the logarithms,

$$\therefore \cdot 0000052 : \cdot 0000023 : 1$$

this gives $\cdot 44$, &c.

and as the number lies between 83805 and 83804

$$\therefore \text{ required number} = 8380 \cdot 444$$

15. *Proportional parts.*

In the preceding example we have seen that the tabular difference is $\cdot 0000052$, or simply 52 by omitting the ciphers, and by referring to the tables we find that there is a difference of 52 between the mantissæ of two consecutive numbers, this 52 is then put in the right-hand column of the tables. If we assume, then, that the difference in the numbers is proportional to the difference in the logarithms, we have

$$\begin{aligned} \text{Difference for } \cdot 1 &= 52 \times \cdot 1 = 5 \\ \text{,, \quad} \cdot 2 &= 52 \times \cdot 2 = 10 \\ \text{,, \quad} \cdot 3 &= 52 \times \cdot 3 = 16 \\ \text{,, \quad} \cdot 4 &= 52 \times \cdot 4 = 21 \\ \text{\&c.} \quad &= \quad \text{\&c.} \end{aligned}$$

And by looking at the bottom of the page we find those numbers 5, 10, 16, 21, &c., placed in a row under the columns headed 1, 2, 3, 4, &c. If we, therefore, required the difference for ·4, we take from this row at the bottom of the page, the number 21, and thus save the trouble of working it in full.

Suppose then that we required to find the logarithm of 838 · 09267,

$$\begin{array}{llr} \text{here log } 838\cdot09 & = & 2\cdot9232907 \\ \text{add for} \quad\quad 2 & & 10 \\ \quad ,, \quad\quad\quad 6 & & 31 \\ \quad ,, \quad\quad\quad 7 & & 36 \\ \hline \therefore \text{ log } 838\cdot09267 & = & 2\cdot923292046 \\ & = & 2\cdot9232920 \end{array}$$

correct to seven places of decimals.

USE OF TRIGONOMETRICAL TABLES.

16. *To find the sine of a given angle.*

If the given angle be found in the tables, the sine, cosine, &c., are easily found by looking in the proper column. If the angle is not found exactly in the tables, we proceed as follows:—

Find the sine of 55° 36′ 35″

$$\begin{array}{rcl} \text{Now sin } 55^\circ\ 36' & = & \cdot8251135 \\ \text{sin } 55^\circ\ 37' & = & \cdot8252778 \\ \hline \text{Diff. for} \quad 60'' & = & \cdot0001643 \end{array}$$

and as the required sine lies between those above, hence the proportion,

$$\begin{array}{rcl} 60'' : 35'' & :: & \cdot0001643 \\ & = & \cdot0000958 \\ \therefore \text{ sin } 55^\circ\ 36'\ 35'' & = & \cdot8251135 + \cdot0000958 \\ & = & \cdot8252093 \end{array}$$

17. *To find the angle corresponding to a given sine.*

If the given sine be found in the tables, it is easy to find

the corresponding angle directly; but if the given sine be not found exactly in the tables, we proceed thus:—

Required the angle corresponding to the given sine ·9962385.

$$\begin{aligned} \sin 85^\circ\, 2' &= \cdot 9962452 \\ \sin 85^\circ\, 1' &= \cdot 9962200 \\ \hline \text{Diff.} &= \cdot 0000252 \end{aligned}$$

The difference between the given sine and sin 85° 1′ = ·0000185, and we see that the required angle must lie between those above; hence the proportion,

$$\cdot 0000252 : \cdot 0000185 :: 60''$$
$$= 34''$$

and therefore the required angle = 85° 1′ 34″

18. *To find the cosine of a given angle.*

If the given angle be not contained exactly in the tables, we proceed as follows:—

Find the cosine of 55° 36′ 35″

$$\begin{aligned} \text{Now } \cos 55^\circ\, 36' &= \cdot 5649670 \\ \cos 55^\circ\, 37' &= \cdot 5647270 \\ \hline \text{Diff. for } 60'' &= \cdot 0002400 \end{aligned}$$

Now this angle 55° 36′ 35″ being in the 1st quadrant, its *cosine* decreases as the angle increases up to 90°, consequently the required cosine will be less than the cosine of 55° 36′, and since the required *cosine* lies between those above, hence the proportion,

$$\begin{aligned} 60'' : 35'' :: \cdot 0002400 &= \cdot 0001400 \\ \text{therefore cosine } 55^\circ\, 36'\, 35'' &= \cdot 5649670 - \cdot 0001400 \\ &= \cdot 5648270 \end{aligned}$$

19. *To find the angle corresponding to a given cosine.*

If the given cosine be not found exactly in the tables, we proceed as follows:—

Required the angle corresponding to the cosine ·5648270.

$$\begin{aligned} \cos 55^\circ\, 36' &= \cdot 5649670 \\ 55^\circ\, 37' &= \cdot 5647270 \\ \hline \text{Diff. for } 60'' &= \cdot 0002400 \end{aligned}$$

The required angle will be between the above angles, and its *cosine* is less than the *cosine* of 55° 36′ by ·5649670 − ·5648270 or ·0001400.

Hence the proportion,

$$\cdot 0002400 : \cdot 0001400 :: 60''$$
$$= 35''$$

Therefore the required angle corresponding to the given cosine ·5648270 is 55° 36′ 35″.

20. In working such questions as the preceding, it must always be borne in mind that in the 1st quadrant, as the angle increases the sine, tangent, and secant increase, while the cosine, cotangent, and cosecant decrease, and *vice versâ*.

21. Since the sine and cosine of an angle are less than unity, their logarithms to the base 10 are negative; and in order to avoid the inconvenience of these negative signs in the tables, we add 10 to the logarithm of all trigonometrical functions before tabulating them: the resulting numbers are called the "tabular logarithms" of the sine of A, cosine of A, &c. These are generally written L. sin A, L. cos A, &c., and are really equal to the real logarithm of sin A, cos A, &c., increased by 10.

The tabular radius is in this case assumed to be 10,000,000,000 instead of 1, and consequently each of the sines, cosines, &c., thus found must be increased in the same proportion.

Thus if s = the natural sine of an arc, and S = the sine of the same arc, whose logarithm is the logarithmic sine,

$$\text{then } S = 10{,}000{,}000{,}000\, s$$
$$\therefore \text{L. } S = \text{L. } 10^{10} + \text{L. } s$$
$$= 10 + \text{L. } s.$$
$$\text{or L. } s = \text{L. } S - 10.$$

Given L. sin $16° \, 24' = 9 \cdot 4507747$, find its natural sine.

$$\text{L. natural sine} = \text{L. sin } 16° \, 24' - 10$$
$$= 9 \cdot 4507747 - 10 = \bar{1} \cdot 4507747$$
$$\therefore \text{natural sine} = \cdot 28234.$$

22. *To find the tabular logarithmic sine of an angle.*

Suppose the given angle is not found exactly in the tables, take the L. sin of the next higher and lower, and proceed as below:—

Find the L. sin $32° \, 18' \, 24'' \cdot 6$

$$\text{Now L. sin } 32° \, 18' \, 20'' = 9 \cdot 7278943$$
$$\text{L. sin } 32° \, 18' \, 30'' = 9 \cdot 7279276$$
$$\text{Diff. for } 10'' = \cdot 0000333$$

and since the necessary L. sin lies between those above, hence the proportion,

$$10'' : 4'' \cdot 6 :: 0000333$$
$$= \cdot 0000153$$
$$\therefore \text{L. sin } 32° \, 18' \, 24'' \cdot 6 = 9 \cdot 7278943 + \cdot 0000153$$
$$= 9 \cdot 7279096.$$

23. *Required the angle corresponding to a given tabular logarithmic sine.*

If the given L. sine be not found exactly in the tables, we take the next higher and lower and the corresponding angles, and proceed as below: –

Find the angle having for its L. sin . $9 \cdot 7279096$

$$\text{L. sin } 32° \, 18' = 9 \cdot 7278277 \text{ next lower}$$
$$\text{L. sin } 32° \, 19' = 9 \cdot 7280275 \text{ next higher}$$
$$\text{Diff. for } 60 = \cdot 0001998$$

and excess of given L. sin over $9 \cdot 7278277$

$$= 9 \cdot 7279096 - 9 \cdot 7278277 = \cdot 0000819$$

Hence the proportion,

$$\cdot 0001998 : \cdot 0000819 :: 60''$$
$$= 24'' \cdot 6$$
$$\therefore \text{angle corresponding to given L. sin is } 32° \, 18' \, 24'' \cdot 6.$$

24. *To find the L. cos of a given angle.*

If the angles be not found exactly in the tables, we proceed as below:—

Find L. cos 32° 18′ 24″·6 from the tables.

$$\begin{aligned}
\text{L. cos } 32^\circ\ 19' &= 9\cdot9269114 \\
\text{L. cos } 32^\circ\ 18' &= 9\cdot9269913 \\
\text{Diff. for } 60'' &= \cdot0000799 \\
\text{and } 60'' : 24''\cdot6 &:: \cdot0000799 \\
&= \cdot0000328 \\
\therefore\ \text{L. cos } 32^\circ\ 18'\ 24''\cdot6 &= 9\cdot9269913 - \cdot0000328 \\
&= 9\cdot9269585.
\end{aligned}$$

25. *To find the angle corresponding to a given tabular logarithmic cosine.*

Find the angle whose L. cos is 9·9269585.

$$\begin{aligned}
\text{L. cos } 32^\circ\ 18' &= 9\cdot9269913 \text{ next higher} \\
\text{L. cos } 32^\circ\ 19' &= 9\cdot9269114 \text{ next lower} \\
\text{Difference for } 60^\circ &= \cdot0000799
\end{aligned}$$

and since the angle falls between these above and 9·9269913 − 9·9269585 = ·0000328, hence the proportion,

·0000799 : ·0000328 :: 60″ = 24″·6.

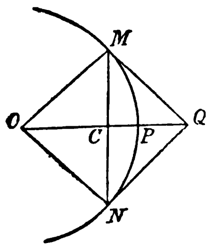
$\therefore$ required angle = 32° 18′ 24″·6

26. In a similar manner we can treat of the other trigonometrical functions, always bearing in mind the fact that L. tan A, L. sec A *increase* as the angle increases (in the 1st quadrant), and that the L. cosec A, L. cot A, *decrease* as the angle increases.

CONSTRUCTION OF TRIGONOMETRICAL TABLES.

27. *The circular measure of an angle less than a right angle is greater than the sine and less than the tangent of the angle, or if θ be the circular measure of an angle, then $\theta > \sin\theta$ and $< \tan\theta$.*

Let the angle MOQ = θ, make $\angle$ QON = $\angle$ MOQ with centre O and any radius, describe a circle cutting



OM, ON, in M and N. From M and N draw tangents meeting in Q. Join MN, meeting OQ in C.

We assume that the line MN < arc MPN, and that the arc MPN < MQ + QN

$$\therefore \text{ MP} > \text{MC and} < \text{MQ}$$

$$\therefore \frac{\text{MP}}{r} > \frac{\text{MC}}{r} \text{ and } < \frac{\text{MQ}}{r}$$

$$\text{or } \theta > \sin\theta \text{ and } < \tan\theta.$$

And since θ lies between $\sin\theta$ and $\tan\theta$

$$\therefore \frac{\theta}{\sin\theta} \text{ lies between 1 and } \frac{1}{\cos\theta}$$

and as θ diminishes indefinitely, $\cos\theta$ increases until when $\theta = 0$, then $\cos\theta = 1$.

Hence as θ diminishes indefinitely $\frac{\theta}{\sin\theta}$ becomes equal to 1, and $\therefore \frac{\sin\theta}{\theta}$ also is equal to 1,

and as $\frac{\tan\theta}{\theta} = \frac{\sin\theta}{\theta} \times \frac{1}{\cos\theta} = 1 \times 1 = 1$

$$\therefore \frac{\tan\theta}{\theta} \text{ also becomes equal to 1.}$$

28. *If θ be less than $90°$, then*

$$\sin\theta > \theta - \frac{\theta^3}{4}.$$

$$\text{since } \sin\theta = 2\sin\frac{\theta}{2}.\cos\frac{\theta}{2} = \frac{2\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}.\cos^2\frac{\theta}{2}$$

$$\text{and since } \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \tan\frac{\theta}{2} > \frac{\theta}{2}$$

$$\therefore \sin\theta > 2\left(\frac{\theta}{2}\right).\cos^2\frac{\theta}{2}$$

$$> \theta.\cos^2\frac{\theta}{2}$$

$$> \theta\left(1 - \sin^2\frac{\theta}{2}\right)$$

$$> \theta\left\{1 - \left(\frac{\theta}{2}\right)^2\right\}$$

$$> \theta - \frac{\theta^3}{4}$$

29. *If an angle be very small, then its sine may be taken as equal to the circular measure.*

By the last article,

$$\sin\theta > \theta - \frac{\theta^3}{4}$$

$$\therefore \theta - \sin\theta < \frac{\theta^3}{4}$$

If the angle be very small, then $\frac{\theta^3}{4}$ may be omitted altogether, as it will become a very small fraction;

$$\therefore \theta = \sin\theta.$$

30. If θ and φ be the circular measures of two very small angles, then we have

$$\frac{\sin\theta}{\sin\varphi} = \frac{\theta}{\varphi}$$

$$\text{or } \frac{\sin 2''}{\sin 1''} = \frac{2}{1} \therefore \sin 2'' = 2\sin 1''$$

$$\text{or } \frac{\sin 3''}{\sin 1''} = \frac{3}{1} \therefore \sin 3'' = 3\sin 1''$$

and generally if n be a very small number,

$$\frac{\sin n''}{\sin 1''} = \frac{n}{1} \therefore \sin n'' = n\sin 1''$$

31. Before closing the chapter it will be of service to the student to become acquainted with the method of

working questions on logarithms, and the solutions of the following ten are therefore given.

(1.) Multiply $231 \cdot 4$ by $5 \cdot 062$

$$\begin{aligned} \text{Now } \log(231 \cdot 4 \times 5 \cdot 062) &= \log 231 \cdot 4 + \log 5 \cdot 062 \\ &= 2 \cdot 3643634 + \cdot 7043221 \\ &= 3 \cdot 0686855 \end{aligned}$$

$$\therefore 231 \cdot 4 \times 5 \cdot 062 = 1171 \cdot 347$$

(2.) Divide $\cdot 6314$ by $\cdot 007241$

$$\log\left(\frac{\cdot 6314}{\cdot 007241}\right) = \log \cdot 6314 - \log \cdot 007241$$

$$= \bar{1} \cdot 8003046 - \bar{3} \cdot 8597985$$

$$\begin{array}{r} \bar{1} \cdot 8003046 \\ \bar{3} \cdot 8597985 \\ \hline 1 \cdot 9405061 \end{array}$$

$$\therefore \cdot 6314 \div \cdot 007241 = 87 \cdot 19792$$

(3.) Find the value of $(1 \cdot 0045)^{365}$

$$\text{Now } \log(1 \cdot 0045)^{365} = 365 \log(1 \cdot 0045)$$

$$\log 1 \cdot 0045 = \cdot 0019499$$

$$\therefore \log(1 \cdot 0045)^{365} = 365 \times \cdot 0019499$$

$$\text{and } \log(365 \times \cdot 0019499) = \log 365 + \log \cdot 0019499$$

$$\begin{aligned} \log 365 &= 2 \cdot 5622929 \\ \log \cdot 0019499 &= \bar{3} \cdot 2900123 \\ \hline &\quad \bar{1} \cdot 8523052 \end{aligned}$$

$$\therefore 365 \times \cdot 0019499 = \cdot 7117135$$

$$\therefore (1 \cdot 0045)^{365} = 5 \cdot 148888.$$

(4.) Find the fourth root of $\cdot 0076542$.

$$\text{Now } \log(\cdot 0076542)^{\frac{1}{4}} = \tfrac{1}{4} \log \cdot 0076542$$

$$\log \cdot 0076542 = \bar{3} \cdot 8838998$$

$$\begin{aligned} \text{and } \bar{3} \cdot 8838998 \div 4 &= \bar{4} + 1 \cdot 8838998 \div 4 \\ &= \bar{1} \cdot 4709749 \end{aligned}$$

$$\therefore (\cdot 0076542)^{\frac{1}{4}} = \cdot 295784.$$

(5.) Find x from the equation,

$$5^x = 20.$$

Now taking log of both sides, we have

$$x \log 5 = \log 20$$

$$\text{or } x \log\left(\frac{10}{2}\right) = \log(10 \times 2)$$

$$x(1 - \log 2) = 1 + \log 2$$

$$\therefore x = \frac{1 \cdot 30103}{\cdot 69897} = 1 \cdot 86$$

This fraction $\frac{1 \cdot 30103}{\cdot 69879}$ can be worked to $1 \cdot 86$ in a similar manner to question 2.

(6.) Find $\log \left\{\frac{32^{\frac{1}{4}} \times 48^{\frac{1}{3}}}{2 \times 27^{\frac{1}{2}}}\right\}^{\frac{1}{2}}$

given $\log 2 = \cdot 30103$ and
$\log 3 = \cdot 4771213$

$$\text{Now } \log \left\{\frac{32^{\frac{1}{4}} \times 48^{\frac{1}{3}}}{2 \times 27^{\frac{1}{2}}}\right\}^{\frac{1}{2}}$$

$$= \tfrac{1}{2}\left\{\log 32^{\frac{1}{4}} + \log 48^{\frac{1}{3}} - \log 27^{\frac{1}{2}} - \log 2\right\}$$

$$= \tfrac{1}{2}\left\{\tfrac{1}{4}\log 32 + \tfrac{1}{3}\log 48 - \tfrac{1}{2}\log 27 - \log 2\right\}$$

$$= \tfrac{1}{2}\left\{\tfrac{1}{4}\log 2^5 + \tfrac{1}{3}\log(3 \times 2^4) - \tfrac{1}{2}\log 3^3 - \log 2\right\}$$

$$= \tfrac{1}{2}\left\{\frac{5 \log 2}{4} + \frac{\log 3 + 4 \log 2}{3} - \frac{3 \log 3}{2} - \log 2\right\}$$

$$= \tfrac{1}{2}\left\{\frac{19 \log 2 - 14 \log 3}{12}\right\} = - \cdot 0400053.$$

(7.) Given $\log 76 \cdot 563 = 1 \cdot 8840189$
$\log 76 \cdot 564 = 1 \cdot 8840246$
find $\log 765 \cdot 6372$.

For a difference of 1 in the numbers we have a difference of $\cdot 0000057$ in the logarithms, hence the proportion,

$$100 : 72 :: \cdot 0000057$$
$$= \cdot 0000041$$

$$\therefore \log 765 \cdot 6372 = 2 + \cdot 8840189 + \cdot 0000041$$
$$= 2 \cdot 8840230.$$

(8.) Given $\log 247 \cdot 63 = 2 \cdot 3938033$
$\log 247 \cdot 64 = 2 \cdot 3938208$
find the number of which the log is $2 \cdot 3938134$.

Now for a difference of $\cdot 0000175$ in logs we have a difference of 1 in the numbers; what number will correspond to the difference between $\cdot 3938134$ and $\cdot 3938033$ or $\cdot 0000101$, hence the proportion,

$$175 : 101 :: 1 = \cdot 57$$

Therefore the number corresponding to the given logarithm $= 247 \cdot 6357$.

(9.) Given $\text{L. cot } 34° 5' = 10 \cdot 1696508$
$\text{L. cot } 34° 6' = 10 \cdot 1693787$
find A from the equation,

$$\text{L. cot A} = 10 \cdot 1694531.$$

Here difference for $60'' = \cdot 0002721$; what angle corresponds to the difference $10 \cdot 1696508 - 10 \cdot 1694531$ or $\cdot 0001977$

$$\therefore 2721 : 1977 :: 60'' = 43'' \cdot 5$$
$$\therefore A = 34° 5' 43'' \cdot 5.$$

(10.) Given $\text{L. sin } 59° 37' 40'' = 9 \cdot 9358894$
difference for $10'' = \cdot 0000124$
find A from the equation,

$$\text{L. sin A} = 9 \cdot 9358921.$$

Here we have the proportion,

$$124 : 27 :: 10'' = 2'' \cdot 18$$
$$\therefore A = 59° 37' 42'' \cdot 18.$$

EXERCISES.

1. Find $\log \sqrt[3]{\cdot 0020736}$, given $\log 2 = \cdot 30103$ and $\log 3 = \cdot 4771213$.

2. Find the following logarithms :—

$$\log_{10} \cdot 0001, \ \log_{\sqrt{3}} 243 \sqrt[3]{9}, \ \log \cdot 01, \ \log_{\cdot 01} 100.$$

3. Find the characteristics of $\log_6 725$, $\log_{12} 1593$.

4. Show that the logarithm of the base is unity, also that

$$\log_a N = \log_a b \cdot \log_b N$$

and $$\log_b a \times \log_a b = 1.$$

5. How is the characteristic of the logarithm of a number determined?

6. If log N to base $a = P$, find N.

7. If $\log_a 32768 = -5$, find a.

8. If $\log_a 49 = \frac{2}{3}$, find a.

9. If $\log_a 6561 = 8$, find a.

10. Given $\log 3 = \cdot 4771213$
$\log 7 = \cdot 8450980$
$\log 11 = 1 \cdot 0413927$
find $\log \cdot 256'$ and $\log \frac{3}{539}$.

11. Find $\log \left\{ \dfrac{a^n (a^2 - x^2)^{-p}}{(b + x)^{-q}} \right\}^{\frac{1}{m}}$

12. If $x^a + y^b = z$

$x^a - y^b = v$, show that

$$a = \frac{\log \frac{z + v}{2}}{\log x}$$

13. If $\log \frac{a}{b} = \log n$, then

$$\log (a + b) = \log a + \log (1 + \frac{1}{n})$$

$$\log (a - b) = \log a - \log \frac{n}{n - 1}$$

14. What do you mean by the base of a logarithm? Show that, whatever base be selected, the logarithm of the product or quotient of two quantities is equal to the sum or difference of their logarithms. Find log 2 to the base $\frac{1}{1000}$ and to the base π.

Given $\log 2 = \cdot 30103$
$\log \pi = \cdot 49715$

15. Verify the approximate formula,

$$2 \log x = \log (x^2 - 1) + \frac{1}{2x} \log \left(\frac{x+1}{x-1}\right) \left\{1 + \frac{1}{6x} + \frac{1}{90x^4}\right\}$$

upon the following values.

$$x = 1000$$

$$\log 999 = 2 \cdot 99956 \ 54882 \ 25982 \ 31$$

$$\log 1001 = 3 \cdot 00043 \ 40774 \ 79318 \ 64$$

16. Work the following questions by logarithms :—

(a) In what time will a sum of money double itself at 10 per cent. per annum compound interest ?

(b) Find value of

$$\frac{(19 \cdot 4)^3 \times (\cdot 0375)^{\frac{1}{3}}}{(\cdot 72)^{\frac{1}{5}} \times (3607)^{\frac{1}{4}}}$$

17. Explain the words, logarithm, base, characteristic, arithmetical complement. Find the common logarithms of $\frac{3}{8}$, $1\frac{1}{4}$, $6\frac{2}{3}$, $\frac{1}{16}$, 64, $\cdot 0125$, $2 \cdot 25$.

18 Given $\log 2 = \cdot 30103$
$\log 405 = 2 \cdot 607455$; find $\log \cdot 0003$.

19. Given $\log 2 = \cdot 30103$.
$\log 3 = \cdot 4771213$; find $\log 7 \cdot 2$.

20. Find the number whose log is $\bar{2} \cdot 7901693$.
Given $\log 61683 = 4 \cdot 7901655$
$\log 616 \cdot 84 = 2 \cdot 7901725$.

21. Given $\log 2 = \cdot 30103$

$$\log 7 = \cdot 845098; \text{ find } \log \sqrt{\frac{4}{343}}$$

$$\text{also find } \log_{1000} \sqrt{\frac{4}{343}}$$

22. Given $\log 24 = 1 \cdot 38021$
$\log 25 = 1 \cdot 39794$
$\log 26 = 1 \cdot 41497$
find log 117 and log 156.

23. Given $\log 3 \cdot 409 = \cdot 532627$
$\log 3 \cdot 410 = \cdot 532754$
find $\log 34 \cdot 0926$.

24. Find θ from the equations,

(a) $4 \sin \theta \cdot \sin (\theta - a) = 2 \cos a - 1$

(b) .... $\sin \theta + \sin 3 \theta = \sin 2 \theta + \sin 4 \theta$

25. Solve the following equations :—

(a) $\cos \phi \sqrt{m^2 - x^2} + m \sin a = x \sin \phi$

(b) $\cos (x + 1\frac{1}{2}) a + \cos (x + \frac{1}{2}) a = \sin a$.

26. Solve the equations,

$$(a)\quad a^{3x} \cdot b^{4-2x} = c^{2x}$$

$$(b)\quad (2 \cdot 5)^{x} = \frac{25}{4}$$

$$(c)\quad \left.\begin{aligned} \frac{2^{2x}}{2^{x+y}} &= 2^{3} \\ x - 3y &= 0 \end{aligned}\right\}$$

27. Prove the following :—

(a) $\sin A = \sin(36° + A) + \sin 72° - A) - \sin(36° - A) - \sin(72° + A)$

(b) $\cos A = \sin(54° + A) + \sin(54° - A) - \sin(18° + A) - \sin(18° - A)$

28. Show how to find L. sin 1′.

29. Show that $\sin 10'' = \cdot 000048481368$
$\cos 10'' = \cdot 9999999988248.$

30. Find the value of $\dfrac{1}{27^{\frac{1}{3}}}$

Given $\log 2 \cdot 7 = \cdot 4313638$
$\log 5 \cdot 172818 = \cdot 7137272.$

31. What are the characteristics of $\log_3 5$, and $\log_2 7$?

32. Given $\log 2 = \cdot 30103$, find the number of digits in 2^{64}.

33. If N be the number of integers whose logarithms have n for characteristic, and M the number of the integers the logarithms of whose reciprocals have $-m$ for characteristic, prove then that

$$\log N - \log M = n - m + 1$$

34. Show that

$$\log(\tan 2^{n}\theta) = 2\log(2\sin 2^{n}\theta) - \log(2\sin 2^{n+1}\theta)$$

35. Find $\log\left(\dfrac{7}{100}\right)^{\frac{1}{5}}$

Given $\log 7 = \cdot 8450980$
$\log 58751 = 4 \cdot 7690153$
$\log 58752 = 4 \cdot 7690227$

36. Given L. cot 72° 15′ = 9·5052891
L. cot 72° 16′ = 9·5048538
find L. cot 72° 15′ 35″

37. Given L. sin 32° 18′ = 9·7278277 diff. for 60″ = ·0001998
L. cos 32° 18′ = 9·9269913 diff. for 60″ = ·0000799
find L. cos and L. tan 32° 18′ 24″·6.

38. Given L. sin A 30° 21′ = 9·7035329
L. sin A 30° 22′ = 9·7037486
find A from the equation,
L. sin A = 9·7036421

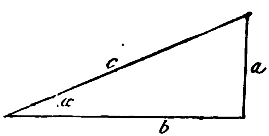
39. Show that $\log_{\tan\theta} \cot\theta = -1$.

40. Given L. cosec 41° 26′ = 10·1793073
L. cosec 41° 27′ = 10·1791642
find L. cosec 41° 26′ 30″

CHAPTER VIII.

RELATIONS BETWEEN THE SIDES AND ANGLES OF A PLANE TRIANGLE.

1. In the annexed right-angled triangle, by our definitions we know that



$$\sin \alpha = \frac{a}{c} \quad \therefore a = c.\sin\alpha \quad . . (1)$$

$$\cos \alpha = \frac{b}{c} \quad \therefore b = c.\cos\alpha \quad . . (2)$$

$$\tan \alpha = \frac{a}{b} \quad \therefore a = b.\tan\alpha \quad . . (3)$$

$$\cot \alpha = \frac{b}{a} \quad \therefore b = a.\cot\alpha \quad . . (4)$$

Hence we get the following simple rules :—

„ (1) Either side is equal to the hypoth. × sin opposite angle.

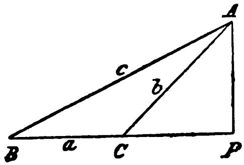
„ (2) Either side is equal to the hypoth. × cos included angle.

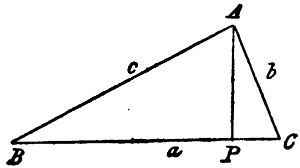
„ (3) Either side is equal to the other × tangent opposite angle.

„ (4) Either side is equal to the other × cotangent adjacent angle.

2. *To show that the sides of a triangle are proportional to the sines of the opposite angles.*

In the annexed triangle ABC, let fall AP perpendicular to BC or BC produced. Now,





$$\left.\begin{aligned} \frac{\mathrm{AP}}{c} &= \sin \mathrm{B} \\ \frac{\mathrm{AP}}{b} &= \sin \mathrm{C} \end{aligned}\right\} \therefore \text{ by division } \frac{b}{c} = \frac{\sin \mathrm{B}}{\sin \mathrm{C}}$$

Similarly it may be proved that $\frac{a}{b} = \frac{\sin \mathrm{A}}{\sin \mathrm{B}}$ and $\frac{a}{c} = \frac{\sin \mathrm{A}}{\sin \mathrm{C}}$

Hence $a : b : c :: \sin \mathrm{A} : \sin \mathrm{B} : \sin \mathrm{C}$.

It may be written in another manner, as

$$\frac{\sin \mathrm{A}}{a} = \frac{\sin \mathrm{B}}{b} = \frac{\sin \mathrm{C}}{c}.$$

3. *To express cos* A, *cos* B, *cos* C *in terms of the sides.*

In the last figure by Euclid (II. 13),

$$b^2 = a^2 + c^2 - 2\,a\,.\,\mathrm{BP}$$

$$\text{but } \mathrm{BP} = c\,.\cos \mathrm{B}$$

$$\therefore b^2 = a^2 + c^2 - 2\,ac\,.\cos \mathrm{B}$$

$$\text{Similarly } a^2 = b^2 + c^2 - 2\,bc\,.\cos \mathrm{A}$$

$$c^2 = a^2 + b^2 - 2\,ab\,.\cos \mathrm{C}$$

or by transposition and division,

$$\cos \mathrm{A} = \frac{b^2 + c^2 - a^2}{2\,bc}$$

$$\cos \mathrm{B} = \frac{a^2 + c^2 - b^2}{2\,ac}$$

$$\cos \mathrm{C} = \frac{a^2 + b^2 - c^2}{2\,ab}$$

4. *To express* $\sin\frac{A}{2}$, $\cos\frac{A}{2}$, $\tan\frac{A}{2}$ *in terms of the sides.*

$$\text{Now } 2\sin^2\frac{A}{2} = 1 - \cos A$$

$$= 1 - \frac{b^2 + c^2 - a^2}{2\,bc} = \frac{2\,bc - b^2 - c^2 + a^2}{2\,bc}$$

$$= \frac{a^2 - (b - c)^2}{2\,bc} = \frac{(a + b - c)\cdot(a + c - b)}{2\,bc}$$

Now let $2\,s = a + b + c \quad \therefore \quad s = \dfrac{a + b + c}{2}$

$$\therefore\ 2\,(s - a) = b + c - a \quad \therefore\ s - a = \frac{b + c - a}{2}$$

$$2\,(s - b) = a + c - b \quad \therefore\ s - b = \frac{a + c - b}{2}$$

$$2\,(s - c) = a + b - c \quad \therefore\ s - c = \frac{a + b - c}{2}$$

$$\text{Hence } 2\sin^2\frac{A}{2} = \frac{2\,(s - b).2\,(s - c)}{2\,bc}$$

$$\therefore\ \sin\frac{A}{2} = \sqrt{\frac{(s - b)\,(s - c)}{bc}}$$

$$\text{Similarly } \sin\frac{B}{2} = \sqrt{\frac{(s - a)\,(s - c)}{ac}}$$

$$\sin\frac{C}{2} = \sqrt{\frac{(s - a)\,(s - b)}{ab}}$$

Again, since

$$2\cos^2\frac{A}{2} = 1 + \cos A$$

$$= 1 + \frac{b^2 + c^2 - a^2}{2\,bc} = \frac{2\,bc + b^2 + c^2 - a^2}{2\,bc}$$

$$= \frac{(b + c)^2 - a^2}{2\,bc} = \frac{(b + c + a)\,(b + c - a)}{2\,bc}$$

$$= \frac{2\,s\,.\,2\,(s - a)}{2\,bc}$$

$$\therefore\ \cos\frac{A}{2} = \sqrt{\frac{s\,(s - a)}{bc}}$$

$$\text{Similarly } \cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ac}}$$

$$\cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$$

$$\tan\frac{A}{2} = \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

$$\tan\frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$$

$$\tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

These formulæ are not difficult, and should be carefully committed to memory.

5. *To find sin* A, *&c., in terms of the sides.*

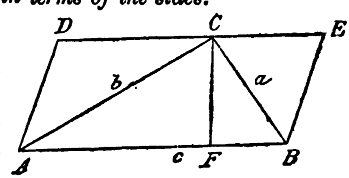
$$\sin A = 2\sin\frac{A}{2}\cdot\cos\frac{A}{2}$$

$$= 2\cdot\sqrt{\frac{(s-b)(s-c)}{bc}}\cdot\sqrt{\frac{s(s-a)}{bc}}$$

$$= 2\sqrt{\frac{s(s-a)(s-b)(s-c)}{bc}}$$

6. *Area of a triangle in terms of the sides.*

Let ABED be any parallelogram, and ABC a triangle on the same base, and between the same parallels; then the area of this triangle is half that of the parallelogram.



$$\text{But } \triangle \text{ parallelogram} = AB\,.\,CF$$

$$\therefore\ \triangle\ ABC = \tfrac{1}{2}\,.\,AB\,.\,CF$$

$$\text{but } CF = b\,.\,\sin A \text{ or } = a\sin B$$

$$\therefore\ \triangle\ ABC = \tfrac{1}{2}\,bc\sin A = \tfrac{1}{2}\,ac\,.\,\sin B$$

Hence the area of any triangle is equal to half the product of any two sides × sin of the included angle.

Now $\triangle$ ABC $= \frac{1}{2}\, bc \,.\, \sin A$

$$= \tfrac{1}{2}\, bc \;.\; 2\sqrt{s\,(s-a)\,(s-b)\,(s-c)}$$

$$\therefore \triangle\ \text{ABC} = \sqrt{s\,(s-a)\,(s-b)\,(s-c)}$$

7. *In every triangle either side is equal to the sum of the product of each of the others into the cosine of the angle which it makes with the first side.*

In this last triangle we have

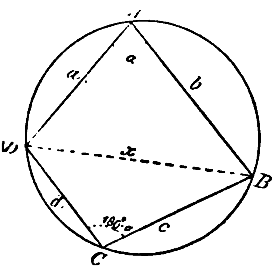
$$c = \text{AF} + \text{FB}$$

$$\therefore c = b \,.\, \cos A + a \,.\, \cos B$$

Similarly $b = a \,.\, \cos C + c \,.\, \cos A$

$$a = b \,.\, \cos C + c \,.\, \cos B.$$

8. *Area of a quadrilateral figure inscribed in a circle in terms of the sides.*



Now let

$$2\,s = a + b + c + d$$

$$\therefore 2\,(s-a) = b + c + d - a$$

$$2\,(s-b) = a + c + d - b$$

$$2\,(s-c) = a + b + d - c$$

$$2\,(s-d) = a + b + c - d$$

The opposite angles A and C are supplementary; draw the diagonal DB and call it x.

Now $\triangle$ ABCD $=$ $\triangle$ ADB $+$ $\triangle$ DBC

$$= \tfrac{1}{2}\, ab \,.\, \sin a + \tfrac{1}{2}\, dc \,.\, \sin\,(180^\circ - a)$$

$$= \tfrac{1}{2}\, ab \,.\, \sin a + \tfrac{1}{2}\, cd \,.\, \sin a$$

$$= \tfrac{1}{2}\,(ab + cd)\,\sin a$$

Again, $x^2 = a^2 + b^2 - 2\,ab \,.\, \cos a$

$$x^2 = c^2 + d^2 + 2\,cd \,.\, \cos a$$

since $\cos\,(180^\circ - a) = -\cos a$

$$\therefore o = a^2 + b^2 - c^2 - d^2 - 2\cos a\,(ab + cd)$$

$$\therefore \cos a = \frac{a^2 + b^2 - c^2 - d^2}{2\,(ab + cd)}$$

$$\text{and } \sin^2 a = 1 - \cos^2 a = 1 - \left\{\frac{a^2 + b^2 - c^2 - d^2}{2(ab + cd)}\right\}^2$$

$$= \frac{4(ab + cd)^2 - (a^2 + b^2 - c^2 - d^2)^2}{4(ab + cd)^2}$$

$$= \frac{\{2(ab + cd) + a^2 + b^2 - c^2 - d^2\}\{2(ab + cd) - a^2 - b^2 + c^2 + d^2\}}{4(ab + cd)^2}$$

$$= \frac{\{(a + b)^2 - (c - d)^2\}\{(c + d)^2 - (a - b)^2\}}{4(ab + cd)^2}$$

$$= \frac{(a + b + c - d)(a + b + d - c)(a + c + d - b)(b + c + d - a)}{4(ab + cd)^2}$$

$$= \frac{2(s - a)\,2(s - b).2(s - c).2(s - d)}{4(ab + cd)^2}$$

$$\therefore \sin a = \frac{2\sqrt{(s - a)(s - b)(s - c)(s - d)}}{(ab + cd)}$$

$$\therefore \triangle \text{ ABCD} = \tfrac{1}{2}(ab + cd).\frac{2\sqrt{(s - a)(s - b)(s - c)(s - d)}}{(ab + cd)}$$

$$= \sqrt{(s - a)(s - b)(s - c)(s - d)}$$

Illustrative Examples.

(1.) If A and C be the greatest and least angles of a triangle of which $a + c = 2b$, then 4 vers A vers C $= (1 - \text{vers A}) + (1 - \text{vers C})$

$$\text{Now } \frac{a}{b} = \frac{\sin A}{\sin B} \cdot \frac{c}{b} = \frac{\sin C}{\sin B}$$

$$\therefore \frac{a + c}{b} = \frac{\sin A + \sin C}{\sin(A + C)} = \frac{2\sin\frac{A + C}{2}.\cos\frac{A - C}{2}}{2\sin\frac{A + C}{2}.\cos\frac{A + C}{2}}$$

$$\therefore \frac{a + c}{b} = \frac{\cos\frac{A - C}{2}}{\cos\frac{A + C}{2}}$$

$$\text{or } \frac{2b}{b} = \frac{\cos\frac{A-C}{2}}{\cos\frac{A+C}{2}}$$

$$\therefore 2\cos\frac{A+C}{2} = \cos\frac{A-C}{2} \ldots (\alpha) \times \text{ each by } \cos\frac{A-C}{2}$$

$$\therefore 2\cos\frac{A+C}{2}.\cos\frac{A+C}{2} = \cos^2\frac{A-C}{2}$$

$$\text{or } \cos A + \cos C = \cos^2\frac{A-C}{2}$$

$$\text{or } (1 - \text{vers } A) + (1 - \text{vers } C) = \cos^2\frac{A-C}{2}$$

Now from (α)

$$\cos\frac{A+C}{2} = \cos\frac{A-C}{2} - \cos\frac{A+C}{2}$$

$$= 2\sin\frac{A}{2}.\sin\frac{C}{2}$$

$$\therefore 2\cos\frac{A+C}{2} = 4\sin\frac{A}{2}.\sin\frac{C}{2}$$

$$\therefore \left(2\cos\frac{A+C}{2}\right)^2 = 16\sin^2\frac{A}{2}.\sin^2\frac{C}{2}$$

$$\text{or } \cos^2\frac{A-C}{2} = 16\sin^2\frac{A}{2}.\sin^2\frac{C}{2}$$

$$= 4 \times 2\sin^2\frac{A}{2}.2\sin^2\frac{C}{2}$$

$$= 4(1 - \cos A)(1 - \cos C)$$

$$= 4 \text{ vers } A . \text{vers } C$$

$$\text{Hence } 4 \text{ vers } A.\text{vers } C = (1 - \text{vers } A) + (1 - \text{vers } C)$$

(2.) In every triangle

$$\cot A + \cot B + \cot C > \tfrac{1}{2}(\text{cosec } A + \text{cosec } B + \text{cosec } C)$$

The expression on the left-hand side

$$= \frac{\cos A}{\sin A} + \frac{\cos B}{\sin B} + \frac{\cos C}{\sin C}$$

$$= \frac{bc \cos A}{2\,\Delta} + \frac{ac \cos B}{2\,\Delta} + \frac{ab \cos C}{2\,\Delta}$$

since $\Delta = \dfrac{bc \sin A}{2} = \dfrac{ab \sin C}{2} = \dfrac{ac \sin B}{2}$

Hence the above will be true if this last expression be greater than $\frac{1}{2}\left(\dfrac{1}{\sin A} + \dfrac{1}{\sin B} + \dfrac{1}{\sin C}\right)$

i.e., if $\dfrac{bc \cos A + ac \cos B + ab \cos C}{2\,\Delta} > \frac{1}{2}\left(\dfrac{bc + ac + ab}{2\,\Delta}\right)$

" $2\,bc \cos A + 2\,ac \cos B + 2\,ab \cos C > ab + bc + ac$

" $b^2 + c^2 - a^2 + a^2 + c^2 - b^2 + a^2 + b^2 - c^2 > ab + bc + ac$

" $a^2 + b^2 + c^2 > ab + bc + ac$

and this is the case, for

$$a^2 + b^2 = 2\,ab + (a - b)^2$$
$$a^2 + c^2 = 2\,ac + (c - a)^2$$
$$b^2 + c^2 = 2\,bc + (b - c)^2$$

$\therefore 2\,(a^2 + b^2 + c^2) = 2\,(ab + bc + ca) + (a - b)^2 + (c - a)^2 + (b - c)^2$

$\therefore a^2 + b^2 + c^2 = a\,b + b\,c + c\,a +$ *another quantity*,

Hence $a^2 + b^2 + c^2 > a\,b + b\,c + c\,a$, and, therefore,
$\cot A + \cot B + \cot C > \frac{1}{2}\,(\operatorname{cosec} A + \operatorname{cosec} B + \operatorname{cosec} C)$

(3.) If ABC be the angles of a triangle, and
$\tan A : \tan B : \tan C :: \sqrt{x} : \sqrt{y} : \sqrt{x} + \sqrt{y}$
then $\tan A \,.\, \tan B = 2$

Now $\tan A + \tan B : \tan A :: \sqrt{x} + \sqrt{y} : \sqrt{x}$

and $\tan C : \tan A :: \sqrt{x} + \sqrt{y} : \sqrt{x}$

$$\therefore \tan A + \tan B = \tan C$$
$$= -\tan(A + B)$$
$$= \frac{-(\tan A + \tan B)}{1 - \tan A \tan B}$$

$$\therefore\ 1 = \frac{-1}{1 - \tan A \tan B}$$

$$1 - \tan A \tan B = 2$$

$$\therefore\ \tan A \tan B = 2$$

(4.) In every triangle

$$4 \sin A \,.\, \cos B \,.\, \cos C = \sin 2B + \sin 2C - \sin 2A$$

Now $\sin 2B + \sin 2C - \sin 2A$

$$\begin{aligned}
&= 2 \sin (B + C) \,.\, \cos (B - C) - \sin 2A \\
&= 2 \sin A \,.\, \cos (B - C) - 2 \sin A \,.\, \cos A \\
&= 2 \sin A \{\cos (B - C) + \cos (B + C)\} \\
&= 2 \sin A \,.\, 2 \cos B \,.\, \cos C \\
&= 4 \sin A \,.\, \cos B \,.\, \cos C.
\end{aligned}$$

(5.) If $\cos A = \dfrac{a}{b + c}$, $\cos B = \dfrac{b}{a + c}$, $\cos C = \dfrac{c}{a + b}$

then $\tan^2 \dfrac{A}{2} + \tan^2 \dfrac{B}{2} + \tan^2 \dfrac{C}{2} = 1.$

Now $2 \sin^2 \dfrac{A}{2} = 1 - \cos A = 1 - \dfrac{a}{b + c} = \dfrac{b + c - a}{b + c}$

and $2 \cos^2 \dfrac{A}{2} = 1 + \cos A = 1 + \dfrac{a}{b + c} = \dfrac{a + b + c}{b + c}$

$$\therefore\ \tan^2 \frac{A}{2} = \frac{b + c - a}{b + c + a} = \frac{s - a}{s}$$

also $\tan^2 \dfrac{B}{2} = \dfrac{s - b}{s}$

and $\tan^2 \dfrac{C}{2} = \dfrac{s - c}{s}$

$$\therefore\ \tan^2 \frac{A}{2} + \tan^2 \frac{B}{2} + \tan^2 \frac{C}{2} = \frac{3s - (a + b + c)}{s}$$

$$= \frac{3s - 2s}{s} = 1$$

EXERCISES.

1. In any triangle

$$\frac{a^2 - b^2}{c^2}\sin C + \frac{b^2 - c^2}{a^2}\sin A + \frac{c^2 - a^2}{b^2}.\sin B +$$

$$4\sin\frac{A - B}{2}.\sin\frac{B - C}{2}.\sin\frac{C - A}{2} = 0$$

2. If $\cot\frac{A}{2} = 1$, $\cot\frac{B}{2} = 2$, $\cot\frac{C}{2} = 3$

then the triangle is right-angled, and $c : b : a :: 3 : 4 : 5$.

Now since $\cot\frac{A}{2} = 1$

$$\therefore\ \frac{A}{2} = 45^\circ\ \therefore\ A = 90^\circ$$

and, therefore, the triangle is right-angled. The remainder is simple.

3. If sin A, sin B, sin C are in harmonical progression, then vers A, vers B, vers C are also.

4. In a triangle ABC . where $C = 90^\circ$, prove that

$$\tan 2A - \sec 2B = \frac{b + a}{b - a}$$

5. If

$$\cos A = \cos B . \cos C$$
$$\cos B = \cos A . \cos C$$
$$\cos C = \cos A . \cos B \text{ . prove that}$$
$$\cos^2 A + \cos^2 B + \cos^2 C = 1.$$

6. Find $\sin\alpha$ from the equation, $\tan\alpha + \sec\alpha = a$.

7. In a triangle, if $a\cos B + b\sin A = c$, find $\tan\frac{A}{2}$

8. If $A + B + C = (2 \pm 1)\frac{\pi}{2}$

then $\sin^2 A + \sin^2 B + \sin^2 C = 1 \pm 2\sin A . \sin B . \sin C$.

9. If $\cos\alpha = \dfrac{\cos\beta - m}{1 - m\cos\beta}$, find $\tan\frac{\alpha}{2}$

10. If $\dfrac{\sin C}{\sin B} = \dfrac{\sin(A + C)}{\sin(A - B)}$, show that

$$\cot B - \cot C = \cot(A + C) + \cot(A - B).$$

11. Find an expression for the sum of the perpendiculars from the angles of a triangle upon the opposite sides.

12. Find an expression for the perpendicular from the angle C on the opposite side c.

13. If two circles touch externally, and r, r^1 denote their radii, prove that $\sin a = \dfrac{4(r - r^1)\sqrt{rr^1}}{(r + r^1)^2}$
where a is the angle between their common tangents.

14. In any triangle prove that

$$ab \cos C + ac \cos B + bc \cos A = \frac{a^2 + b^2 + c^2}{2}$$

$$\text{and } \frac{a}{s - a} = \frac{\cot\frac{B}{2} + \cot\frac{C}{2}}{\cot\frac{A}{2}}$$

15. Prove that

$$a^2 + b^2 + c^2 = 2\,ab \cos C + 2\,bc \cos A + 2\,ac \cos B.$$

16. In any triangle

$$a + b + c = 2c \cdot \frac{\cos\frac{A}{2} \cdot \cos\frac{B}{2}}{\cos\frac{A + B}{2}}$$

17. If $A + B + C = 180°$, prove that

$$2s = (a + b) \cos C + (b + c) \cos A + (a + c) \cos B.$$

18. In any plane triangle show that

$$\frac{\tan\frac{A}{2}}{\tan\frac{C}{2}} = \frac{1 - \cos A + \cos B + \cos C}{1 - \cos C + \cos A + \cos B}$$

19. If $A + B + C + D = 2\pi$, prove that

$$\tan A \cdot \tan B \cdot \tan C \cdot \tan D = \frac{\tan A + \tan B + \tan C + \tan D}{\cot A + \cot B + \cot C + \cot D}$$

20. In the preceding example, if

$$\frac{\tan B}{\tan A} = \frac{\tan D}{\tan C}, \text{ then}$$

$$\tan A \tan D = \tan B \cdot \tan C = 1.$$

21. Expand $\tan(A + B + C + D)$ if $A + B + C + D = 2\pi$.

22. If $a : b : c :: \cos A : \cos B : \cos C$

$$\text{and } A + B + C = 90°$$

$$\text{then } \sin A = \frac{b^2 + c^2 - a^2}{2\,bc}$$

23. Show that $\dfrac{\sin A + \sin B + \sin C}{\sin A + \sin B - \sin C} = \cot\dfrac{A}{2} . \cot\dfrac{B}{2}$

if $A + B + C = 180°$.

24. Eliminate C and c from the following equations:—

$$a = b \cos C + c \cos B$$
$$b = c \cos A + a \cos C$$
$$c = a \cos B + b \cos A.$$

25. Show that

$$(\sin \alpha + \cos \alpha)^3 = \tfrac{3}{2}(\sin \alpha + \cos \alpha) + \tfrac{1}{2}(\sin 3\alpha - \cos 3\alpha).$$

26. If $\dfrac{\cot \dfrac{C}{2}}{\cot \dfrac{a}{2}} = \tan \dfrac{3\pi - 2\beta}{4}$, show that

$$\frac{\cos \alpha + \sin \beta}{\sin \alpha \cos \beta} = \cot (\pi - C)$$

27. If $a : b : c :: \cos \alpha : \cos 2\alpha : \cos 3\alpha$

then $\cos \alpha = \dfrac{a + c}{2b}$

28. If $\cot A + \cot C = 2 \cot B$

then $a^2 + c^2 = 2b^2$

29. In every triangle show that

$$\cos A + \cos B + \cos C = 1 + \frac{a \sin B . \sin C}{s}$$

30. If $\dfrac{\cos(\theta + \alpha)}{\cos(\theta - \alpha)} = \dfrac{\cos\theta - \cos(\theta + \alpha)}{\cos(\theta - \alpha) - \cos\theta}$, find $\cos\theta$

31. If $\left.\begin{aligned}\frac{m}{x} &= \cos \alpha + \cos 2\alpha \\ \frac{n}{y} &= \sin \alpha + \sin 2\alpha\end{aligned}\right\}$ show that

$$\left\{\frac{m^2}{x^2} + \frac{n^2}{y^2}\right\}^2 - 3\left\{\frac{m^2}{x^2} + \frac{n^2}{y^2}\right\} - \frac{2m}{x} = 0.$$

32. If in a triangle

$$\frac{1 + 3\sin^2 \dfrac{A}{2}}{1 + 3\sin^2 \dfrac{C}{2}} = \frac{\sin^2 A}{\sin^2 C}$$

the triangle is either isosceles, or has its sides in arithmetical progression.

33. In a triangle,

$$c = \sqrt{(a+b)^2 \sin^2 \frac{C}{2} + (a-b)^2 \cos^2 \frac{C}{2}}$$

$$= \frac{a}{\cos B + \sin B \cot C}$$

34. If $\tan A \,.\, \tan B = 3$

and $\dfrac{\sin A}{\sin B} = \dfrac{a}{b}$

$$\tan 2\phi = \frac{2ab}{3(a^2 - b^2)}$$

then will $\tan A = \sqrt{\dfrac{3a \cot \phi}{b}}$

$$\tan B = \sqrt{\frac{3b \tan \phi}{a}}$$

35. If a perpendicular, p, be let fall from the angle C on the opposite side, then

$$p = \frac{c}{2} \cdot \frac{\cos(A-B) - \cos(A+B)}{\sin C}$$

36. If x be the distance of this perpendicular from the middle of the base, then

$$x = \frac{c}{2} \cdot \frac{\tan A - \tan B}{\tan A + \tan B}$$

$$\text{or } x = \frac{c}{2} \cdot \frac{\sin(A-B)}{\sin C}$$

37. If Δ be the area of a triangle, then

$$\Delta = \frac{2abc}{a+b+c} \cdot \cos \frac{A}{2} \cdot \cos \frac{B}{2} \cdot \cos \frac{C}{2}$$

38. If $2 \cos B = \dfrac{\sin A}{\sin C}$, then the triangle is isosceles.

39. In an oblique-angled triangle prove the following: —

(a) $$\frac{c}{a-b} = \frac{\tan \frac{A}{2} + \tan \frac{B}{2}}{\tan \frac{A}{2} - \tan \frac{B}{2}}$$

(b) $$\frac{\text{vers } A}{\text{vers } B} = \frac{a(s-b)}{b(s-a)}$$

(c) $$\frac{\sin(A-B)}{\sin(A+B)} = \frac{a^2 - b^2}{c^2}$$

(d) $$\frac{\tan \frac{A + B - C}{2}}{\tan \frac{A + C - B}{2}} = \frac{a^2 + b^2 - c^2}{a^2 + c^2 - b^2} = \frac{\tan B}{\tan C}$$

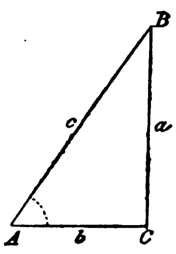
CHAPTER IX.

ON THE SOLUTION OF TRIANGLES.

RIGHT-ANGLED TRIANGLES.

1. Under this head we have different cases, which are as follow:—

(1.) Given hypothenuse and one acute angle.
(2.) Given one side and adjacent angle.
(3.) Given one side and opposite angle.
(4.) Given the two sides.
(5.) Given hypothenuse and one side.



Case 1. Given c and A.

Now $b = c.\cos A$, and $a = c.\sin A$

$\therefore$ by taking the logarithms of each side of these equations,

$$\log b = \log c + \text{L.}\cos A - 10$$
$$\log a = \log c + \text{L.}\sin A - 10$$

which give a and b respectively.

Since the angles A and B are complementary,

$$\therefore B = 90° - A.$$

Case 2. Given b and A.

Now $a = b.\tan A$, and $c = \dfrac{b}{\cos A}$

$$\therefore \log a = \log b + \text{L.}\tan A - 10$$
$$\log c = \log b - \text{L.}\cos A + 10$$
$$B = 90° - A.$$

Case 3. Given a and A.

Now $b = a.\cot A$, and $c = \dfrac{a}{\sin A}$

$$\therefore \log b = \log a + \text{L.} \cot A - 10$$
$$\log c = \log a - \text{L.} \sin A + 10$$
$$B = 90^\circ - A.$$

Case 4. Given a and b.

$$\text{Tan } A = \frac{a}{b}, \text{ and } c = \sqrt{a^2 + b^2}$$
$$\therefore \text{L.} \tan A - 10 = \log a - \log b$$
$$\text{or L.} \tan A = \log a - \log b + 10$$
$$B = 90^\circ - A.$$

Case 5. Given c and b.

Now $\cos A = \dfrac{b}{c}$

$$\therefore \text{L.} \cos A = \log b - \log c + 10$$
$$B = 90^\circ - A$$
$$\text{and } a = \sqrt{c^2 - b^2}$$

This last equation $a = \sqrt{c^2 - b^2}$ might be adapted to logarithmic computation.

$$\begin{aligned} \log a &= \tfrac{1}{2} \log (c^2 - b^2) \\ &= \tfrac{1}{2} \log (c + b)(c - b) \\ &= \tfrac{1}{2} \{ \log (c + b) + \log (c - b) \} \end{aligned}$$

Example, Case 3: Given $a = 379 \cdot 63$

$A = 39^\circ\ 26'\ 15''$ and $C = 90^\circ$

find the other parts.

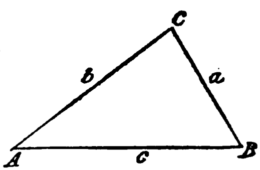
$$\text{Now } B = 90^\circ - A = 20^\circ\ 33'\ 45''$$
$$\frac{a}{b} = \tan A$$
$$\therefore \log a - \log b = \text{L.} \tan A - 10$$
$$\therefore \log b = \log a + 10 - \text{L.} \tan A$$

$$\begin{aligned} &= 2 \cdot 5793605 \\ &\quad\ 10 \\ &\quad\ \overline{12 \cdot 5793605} \\ \text{L.} \tan A &= 9 \cdot 9151390 \\ \therefore \log b &= \overline{2 \cdot 6642215} \\ \therefore b &= 461 \cdot 56 \end{aligned}$$

also $c = \dfrac{a}{\sin A}$ $\therefore \log c = \log a - \text{L.} \sin A + 10$

$$
\begin{array}{r}
= 2 \cdot 5793605 \\
10 \\
\hline
12 \cdot 5793605 \\
9 \cdot 8029352 \\
\hline
2 \cdot 7764253
\end{array}
\therefore c = 597 \cdot 62
$$

OBLIQUE-ANGLED TRIANGLES.



2. *Case* 1. Given two angles and a side opposite one of them.

Case 2. Given two angles and a side adjacent to them.

Case 3. Given two sides and the included angle.

Case 4. Given two sides and the angle opposite one of them.

Case 5. Given three sides.

Case 1. Given A, B, and a.

$$C = 180^\circ - (A + B)$$

$$\frac{a}{b} = \frac{\sin A}{\sin B}, \quad \frac{a}{c} = \frac{\sin A}{\sin C}$$

$$\therefore \log b = \log a + \text{L.} \sin B - \text{L.} \sin A$$
$$\log c = \log a + \text{L.} \sin C - \text{L.} \sin A$$

Case 2. Given A, B, and c.

$$C = 180^\circ - (A + B)$$

$$\frac{a}{c} = \frac{\sin A}{\sin C} \text{ and } \frac{b}{c} = \frac{\sin B}{\sin C}$$

$$\therefore \log a = \log c + \text{L.} \sin A - \text{L.} \sin C$$
$$\log b = \log c + \text{L.} \sin B - \text{L.} \sin C$$

Case 3. Given a, b, and C.

We must find the angles A and B.

$$\text{Now, } \frac{a}{b} = \frac{\sin A}{\sin B}$$

$$\frac{a+b}{a-b} = \frac{\sin A + \sin B}{\sin A - \sin B}$$

$$= \frac{2 \sin \frac{A+B}{2} \cdot \cos \frac{A-B}{2}}{2 \cos \frac{A+B}{2} \cdot \sin \frac{A-B}{2}}$$

$$\therefore \frac{a+b}{a-b} = \frac{\tan \frac{A+B}{2}}{\tan \frac{A-B}{2}}$$

$$\text{And } \tan \frac{A+B}{2} = \tan\left(90^\circ - \frac{C}{2}\right) = \cot \frac{C}{2}$$

$$\therefore \text{ L. } \tan \frac{A-B}{2} = \text{L. } \tan \frac{A+B}{2} + \log(a-b) - \log(a+b)$$

$$= \text{L. } \cot \frac{C}{2} + \log(a-b) - \log(a+b)$$

This finds $\frac{A-B}{2}$; and we know $\frac{A+B}{2}$. Hence, by adding and subtracting, we get A and B respectively.

$$\frac{c}{a} = \frac{\sin C}{\sin A}$$

$$\therefore \log c = \log a + \text{L. } \sin C - \text{L. } \sin A.$$

Hence A, B, and c are determined.

When the difference between a and b is not very great, and the angle C rather large, the following method will give the side c with more exactness, and without having to find the angles A and B beforehand.

We know that

$$c^2 = a^2 + b^2 - 2\,ab \cos C$$

$$= a^2 + b^2 - 2\,ab\left(2 \cos^2 \frac{C}{2} - 1\right)$$

$$= a^2 + b^2 + 2\,ab - 4\,ab \cos^2 \frac{C}{2}$$

$$= (a + b)^2 - 4\,ab \cos^2 \frac{C}{2}$$

$$= (a + b)^2 \left\{ 1 - \frac{4\,ab}{(a + b)^2} \cos^2 \frac{C}{2} \right\}$$

Now $4\,ab > (a + b)^2$ always, and $\cos \frac{C}{2}$ is less than unity;

$$\therefore \frac{4\,ab}{(a + b)^2} . \cos^2 \frac{C}{2} \text{ is a proper fraction,}$$

and some angle can be found the *square of whose sine* is equal to it.

Let α be such an angle, so that

$$\sin^2 \alpha = \frac{4\,ab}{(a + b)^2} . \cos^2 \frac{C}{2} \quad \ldots\ldots\ldots\ldots (1)$$

$$\therefore c^2 = (a + b)^2 (1 - \sin^2 \alpha) = (a + b)^2 . \cos^2 \alpha$$

$$\therefore c = (a + b) \cos \alpha \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

we can find α from (1), thus

$$2\,\text{L}. \sin \alpha = 2 \log 2 + \log a + \log b - 2 \log (a + b) + 2\,\text{L}. \cos \frac{C}{2}$$

hence from (2) we can determine c.

Such an angle as we make use of in (1) is termed a *subsidiary* angle.

Case 4. Given a, b, A,

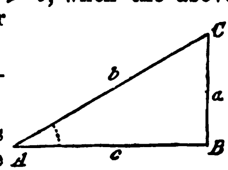
$\sin B = \sin A \dfrac{b}{a}$; this gives the angle B, and hence we determine the angle C:

$$c = b . \frac{\sin C}{\sin A}; \text{ this gives the side } c.$$

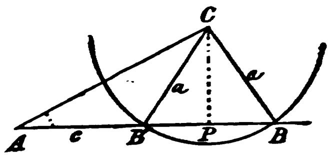
This case is *ambiguous* unless $a > b$, when the above formulæ are all those necessary for the solution.

We will give a geometrical demonstration of this.

If $a = b \sin A$, then there is *only one* solution, and the triangle is then right-angled.



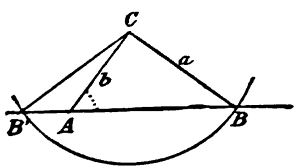
If $a < b \sin A$, then it is evident there could be *no solution* whatever.



If $a < b$, take C as centre and a as radius, and describe a circle, cutting the base-line in B and B'.

Here the *two* triangles ABC, AB'C answer the required conditions, for each has the given parts a, b, and A.

If $a > b$, then describe the circle as before. It is evident that there is only *one* solution here, for the triangle ABC is the only one answering the required conditions—*i.e.*, having the sides a, b, and angle A.



Case 5. Given a, b, c.

Here we have only to find the angles A, B, C, which may be found from either of the following formulæ :—

$$\sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$$

$$\cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$$

$$\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

Similarly for the angles B and C.

Before we finish this chapter, a few examples will be worked by way of illustration.

1. Given $b = 21 \cdot 704$, $c = 17 \cdot 802$, and the included angle $A = 26° \, 12' \, 16''$; find the other parts.

Now $\frac{1}{2}(B + C) = (90° - \frac{A}{2}) = 76° \, 53' \, 52''$; and

$$L. \tan \tfrac{1}{2}(B - C) = L. \tan \tfrac{1}{2}(B + C) + \log(b - c) - \log(b + c)$$

$$\begin{aligned}
\text{L.}\tan\frac{B+C}{2} &= 10\cdot6331137 \\
\log\ 3\cdot902 &= \underline{\ \ 0\cdot5912873} \\
& \quad\ 11\cdot2244010 \\
\log\ 39\cdot506 &= \underline{\ \ 1\cdot5966631} \\
& \quad\ \ \ 9\cdot6277379 \\
\therefore \tfrac{1}{2}(B - C) &= 22^\circ\ 59'\ 41'' \\
\text{and } \tfrac{1}{2}(B + C) &= 76^\circ\ 53'\ 52'' \\
\therefore B &= 99^\circ\ 53'\ 33'' \\
\text{and } C &= 53^\circ\ 54'\ 11''
\end{aligned}$$

To find C, we know that

$$2\text{ L.}\sin a = 2\log 2 + \log b + \log c - 2\log(b+c) + 2\text{ L.}\cos\frac{A}{2}$$

$$\begin{aligned}
2\log 2 &= \ \ \cdot6020600 \\
\log 21\cdot704 &= 1\cdot3365398 \\
\log 17\cdot802 &= 1\cdot2504688 \\
2\text{ L.}\cos 13^\circ\ 6'\ 8 &= \underline{19\cdot9770886} \\
& \quad 23\cdot1661572 \\
2\log 39\cdot506 &= \underline{\ \ 3\cdot1933262} \\
& \quad 19\cdot9728310 \\
\therefore \text{L.}\sin a &= \ \ 9\cdot9864155 \\
\therefore a &= 75^\circ\ 44'\ 37\cdot8''
\end{aligned}$$

Again, $\log a = \log(b + c) + \text{L.}\cos\alpha - 10$

$$\begin{aligned}
\log 39\cdot506 &= \ \ 1\cdot5966631 \\
\text{L.}\cos\alpha &= \underline{\ \ 9\cdot3913900} \\
& \quad 10\cdot9880531 \\
& \quad \underline{10\qquad\qquad} \\
\therefore \log a &= \ \ \ \cdot9880531 \\
\therefore a &= \ \ 9\cdot728
\end{aligned}$$

2. The sides of a triangle are as 4 : 5 : 6; find the angle B.

$$\begin{aligned}
\text{Given } \log 2 &= .3010300 \\
\log 5 &= .6989700 \\
\text{L.}\cos 27^\circ\ 53' &= 9.9464040 \\
\text{L.}\cos 27^\circ\ 54' &= 9.9463371
\end{aligned}$$

$$\text{Now } \cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ac}} = \sqrt{\frac{\frac{15}{2}\times\frac{5}{2}}{24}} = \frac{5}{\sqrt{32}}$$

$$\therefore \text{L.} \cos\frac{B}{2} = 10 + \log 5 - \tfrac{1}{2}\log 32$$

$$= 10 + \log 5 - \tfrac{1}{2}\cdot 5\log 2$$

$$\begin{array}{rr} & 10 \\ \tfrac{1}{2}\,.\,5\log 2 = & \cdot 752575 \\ \hline & 9\cdot 247425 \\ \log 5 = & \cdot 698970 \\ \hline & 9\cdot 946395 \end{array}$$

Now for a difference of $1'$ or $60''$ in the angles we have a corresponding difference of $\cdot 0000669$ in the logarithms. Hence as $\cdot 0000669 : \cdot 0000090 : 60$, this gives $8''$ nearly,

$$\therefore \frac{B}{2} = 27^\circ\ 53'\ 8''$$

$$\therefore B = 55^\circ\ 46'\ 16''$$

3. Given L. sin A $30^\circ\ 21' = 9\cdot 7035329$
L. sin A $30\ \ 22 = 9\cdot 7037486$
find A from the equation,

$$\text{L. sin A} = 9\cdot 7036421$$

Now for a difference of $1'$ or $60''$ in angles there is a difference of $\cdot 0002157$ in logs; for what angle will there be a difference of $9\cdot 7036420 - 9\cdot 7035329$ or $\cdot 0001092$? Hence the proportion,

$$2157 : 1092 :: 60$$

which gives $30''\cdot 3$

$$\therefore A = 30^\circ\ 21'\ 30''\cdot 3$$

4. In the ambiguous case, if x be the value found for the third side in one instance and y the other, prove that

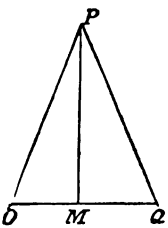
$$x + y = 2b\cos A$$

and $$xy = b^2 - a^2$$

See figure to this in Case 4. Let fall CP perpendicular on AB,

$$\begin{aligned} AB + AB' &= 2\,AB' + B'B \\ &= 2\,AB' + 2\,B'P \\ &= 2\,AP = 2b\cos A \\ \therefore\ x + y &= 2b\cdot\cos A \\ \text{Again, } AP^2 - PB^2 &= b^2 - a^2 \\ \text{or } (AP + PB)(AP - PB) &= b^2 - a^2 \\ \text{or } (AP + PB)(AP - PB') &= b^2 - a^2 \\ \text{or } AB\,.\,AB' &= b^2 - a^2 \\ \therefore\ xy &= b^2 - a^2 \end{aligned}$$

5. The angle at the vertex of an isosceles triangle is 30° and the base is 10 feet. Find the lengths of the equal sides without using logarithms, to three decimal places. (Exam., 1875.)



Let fall PM perpendicular on OQ, then

$$OM = OP\sin 15^\circ$$

$$\therefore\ 5 = OP\sin 15^\circ$$

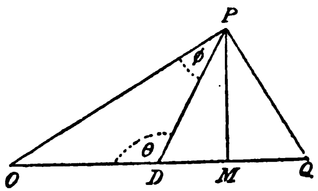
$$\therefore\ OP = \frac{5}{\sin 15} = \frac{5}{\frac{\sqrt{3}-1}{2\sqrt{2}}} = \frac{10\sqrt{2}}{\sqrt{3}-1}$$

$$= \frac{10\sqrt{2}(\sqrt{3}+1)}{2} = 5\sqrt{2}(\sqrt{3}+1)$$

$\therefore\ OP = 19\cdot315$

6. In a triangle OPQ, if PM be let fall perpendicular on OQ, and if the segments OM, MQ be denoted by x and y, prove that

$$\frac{\sin(Q - O)}{\sin(Q + O)} = \frac{x - y}{x + y}$$



Let $OM = x$, $MQ = y$. Take $MD = MQ$, join PD. Let $\angle\ ODP = \theta$, $\angle OPD = \phi$

$$\therefore \angle \varphi = Q - O$$

$$\text{now } OQ = PO \cdot \frac{\sin P}{\sin Q}$$

$$\text{and } OD = PO \cdot \frac{\sin \varphi}{\sin \theta}$$

$$\therefore \frac{OD}{OQ} = \frac{\sin \varphi}{\sin \theta} \times \frac{\sin Q}{\sin P} = \frac{\sin \varphi}{\sin P}$$

$$\text{Since } \sin \theta = \sin Q$$

$$\therefore \frac{x - y}{x + y} = \frac{\sin \varphi}{\sin P} = \frac{\sin (Q - O)}{\sin (Q + O)}$$

Exercises.

1. Two sides of a triangle and the angle opposite one of them being given, show how to find the third side. Under what conditions will the problem admit of two solutions?

2. In a plane triangle, $a = 27$, $b = 19$, $c = 40$. Find B to seconds by means of the formula for $\tan \frac{B}{2}$.

3. The sides of a triangle are 8 feet, 15 feet, and 17 feet. Find the angles to seconds.

4. Establish the working formula for finding the third side of a plane triangle when two sides and included angle are given. What becomes of this formula when the triangle is isosceles?

5. If a subsidiary angle ϕ is employed in the last problem, and the formula to be used is

$$\tan \frac{B - C}{2} = \tan^2 \frac{\phi}{2} \cdot \cot \frac{A}{2}$$

what does ϕ represent?

6. Find the sum of the areas of the two triangles in the ambiguous case.

7. If $a = 210$, $b = 110$, $C = 34° 42' 30''$, show how to find the other angles, having given $\log 2 = \cdot 30103$

$$L. \cot 17° 21' 15'' = 10 \cdot 5051500.$$

8. Find an expression for the area of a triangle in terms of the sides, and by means of it find the area of a triangle whose sides are 16·5, 13, and 11·5 feet respectively.

9. In any plane triangle, given b, c, and angle A, find the side a without finding B and C in a form adapted to logarithmic computation.

10. In a triangle ABC, the angle $A = 63° 44' 40''$, the side $AB = 405 \cdot 91$ feet, and the side $AC = 328 \cdot 27$ feet. Find the area and the ratio in which the side AC is divided by the perpendicular upon it from B.

11. In a level country a base AB is measured on one side of a river, and from each end of it an object P is observed. Find PA and PB. $AB = 652 \cdot 3$ feet, and the angles $PAB = 63° 44' 45'$ and $PBA = 37° 10' 35''$.

12. If in any triangle $a \cos A = b \cos B$, prove that the triangle is either right-angled or isosceles; and also, if $a \cos A = b \cos B = c \cos C$, the triangle is equilateral.

13. The area of a right-angled triangle $= \frac{c^2}{4} \cdot \sin 2B = \frac{a^2}{2} \cdot \tan B$.

14. AB is a horizontal base-line 100 yards long, running from north A to south B. Two points P and Q in the same horizontal plane, and to the eastward are observed with a theodolite, and it is found that

$$\angle PAB = 37° 16' 36'' \quad \angle QAB = 120° 13' 42''$$
$$\angle PBA \quad 110° 18' 18'' \quad \angle QBA = \quad 52° 43' 24''.$$

Find the length and direction of the line PQ, noting that $\angle PAB + \angle QBA = 90°$.

15. Show that in any triangle the sides are proportional to the sines of the opposite angles. One angle of a triangle is 120°, and the sides which contain it are as 4 : 1. Show that the cotangents of the other angles are $3\sqrt{3}$ and $\frac{\sqrt{3}}{2}$.

16. Show how to solve a right-angled triangle, having given a side and an acute angle. A staff at the top of a tower is observed to subtend an angle of 15° by an observer at a distance of a feet from the foot of the tower, and also to subtend the same angle when the observer is at a distance of b feet. Find the height of the staff.

17. If $A + B + C = 180°$, prove that

$$\sin \frac{A}{2} + \sin \frac{B}{2} > \sin \frac{C}{2}.$$

18. Express the tangent of half an angle in terms of the sides.

19. If $5 (\cos A + \cos B) = 4 (1 + \cos A \cos B)$, show that the sides of the triangle are in arithmetical progression.

20. The diagonals of a parallelogram are 18 and 24, and one side is 15; find the area.

21. Two sides of a triangle are 3 and 5, and the included angle is 60°; find the other angles. (Exam., 1876.)

22. It is found by observation that the angle which a flagstaff on

the top of a tower subtends is 38′. The elevation of the tower is 14°; find the observer's distance from the tower, and the height of the tower, supposing the flagstaff 24 feet high.

23. A balloon is observed $418\frac{7}{16}$ feet above the ground, its altitude 34° 15′; find its distance from the observer.

24. Show how to find the dip of the horizon, and prove that

$$\text{Dip expressed in minutes} = 1 \cdot 06\sqrt{h}$$

where h is the height of the eye above the sea.

25. Prove that the distance of the sea horizon in miles is equal to the dip in minutes multiplied by $1\frac{3}{20}$,

$$\text{or that } d = 1\tfrac{3}{20} \cdot \text{dip}.$$

26. Given $a = 62\frac{73}{100}$, $b = 30\frac{3}{25}$, and $A = 107° \; 3' \; 13''$; find the other parts.

27. Prove that the area of a triangle

$$= \tfrac{1}{4}\sqrt{2\,(a^2 b^2 + b^2 c^2 + a^2 c^2) - a^4 - b^4 - c^4}$$

28. In a plane triangle prove the following:—

$$\frac{a}{b + c} = \frac{\cos \frac{1}{2}(B + C)}{\cos \frac{1}{2}(B - C)}$$

$$\frac{a}{b - c} = \frac{\sin \frac{1}{2}(B + C)}{\sin \frac{1}{2}(B - C)}$$

29. In a right-angled triangle, where C is the right angle, given $a = 9{\cdot}84$, and $B = 72° \; 47' \; 9''$; find the other parts.

30. Given $b = 379{\cdot}628$, $B = 39° \; 26' \; 15''$, $C = 90°$; find the other parts.

31. In an oblique-angled triangle find B, A, and b, given $a = 195{\cdot}265$, $c = 203{\cdot}162$, and $C = 45° \; 0' \; 55''$.

32. Given $b = 238{\cdot}1974$, $a = 185{\cdot}3011$, $C = 91° \; 23'$; find the other parts.

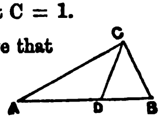
33. If $\dfrac{\tan A}{\tan B} = \dfrac{\sin^2 A}{\sin^2 B}$, prove that $C = 90°$, or that the triangle is isosceles.

34. If $A + B + C = 180°$, prove that

$$\cot A \cot B + \cot B \cot C + \cot A \cot C = 1.$$

35. In a triangle, if CD bisect the angle C, prove that

$$CD = \frac{2\,ab}{a + b} \cdot \cos \tfrac{1}{2} C$$



36. In a triangle, if $c = (a - b) \sec \alpha$, prove then that

$$\tan \alpha = \frac{2\sqrt{ab}}{a - b} \cos \frac{A + B}{2}$$

37. If in the figure to 35 CD bisects the side c, prove that

$$CD = \frac{\sqrt{2(a^2 + b^2) - c^2}}{2}$$

38. In a triangle, if $\cot \frac{A}{2} + \cot \frac{C}{2} = 2 \cot \frac{B}{2}$, then

$$\sin A + \sin C = 2 \sin B.$$

39. In the triangle ABC, AB was found to be 18,735 feet, and AC 20,507 feet. The angle at A was found to be $57° 42' 24''$. Find the area in acres, roods, and perches. (Exam., 1876.)

40. The sides of a triangle are 4, 5, 6; find the angles.

41. If c, the hypothenuse of a right-angled triangle, be trisected in M and N, prove that $CM^2 + CN^2 + MN^2 = \frac{2c^2}{3}$.

42. If the tangents of the angles of a triangle are as 1 : 2 : 3, find them.

43. In a triangle, if $a : c :: b - a : c - b$, prove that

$$\cos \frac{B}{2} = \sqrt{\frac{\sin A \sin C}{\cos A + \cos C}}$$

44. If $A + B + C = 180°$, then

$$8 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} < 1$$

45. Prove $a^2 \sin A + ab \sin B + ac \sin C = (a^2 + b^2 + c^2) \sin A$.

46. In a triangle, if $a^3 + b^3 + c^3 = c^2 (a + b + c)$, and

$$\frac{\sin A}{\sin C} = \frac{\sin C}{\sin B}$$

prove that $a = b = c$.

47. If the bisectors of the angles of a triangle meet the opposite sides, and their lengths be denoted by x, y, z, prove that

$$\frac{\cos \frac{A}{2}}{x} + \frac{\cos \frac{B}{2}}{y} + \frac{\cos \frac{C}{2}}{z} = \frac{ab + ac + bc}{abc}$$

48. A quadrilateral figure in a circle is bisected by one diagonal and trisected by another. If a and c be two adjacent sides which contain an obtuse angle, and are subtended by the trisecting diagonal, show that the area of the quadrilateral or

$$\Delta = \tfrac{1}{4} \sqrt{34 a^2 c^2 - (a^4 + c^4)}$$

49. In the figure to 35, if CD bisect the angle C, and the segments AD and DB be denoted by x and y, and that $x : y :: p : q$; find tan B in terms of A, p, and q.

50. Given $a = 18$, $b = 20$, $c = 22$, find L. tan $\frac{1}{2}$ A.

Given $\log 2 = \cdot 30103$
$\log 3 = \cdot 4771213$.

51. The sides of a triangle subtend equal angles at a point inside it, and the distances of this point from each angular point are p, q, r respectively; prove that

$$a = \sqrt{q^2 + r^2 + qr}$$

$$\text{and } \Delta = \frac{\sqrt{3}}{4}(pq + pr + qr)$$

where Δ denotes the area of the triangle.

52. Given $a = 85{\cdot}63$, $b = 78{\cdot}21$, $C = 48° 24'$; find c, A and B.

Given L. cot 24° 12′ = 10·3473497
L. sin 24° 12′ = 9·6127023
L. tan 5° 45′ = 9·0030066
L. cos 5° 45′ 15″ = 9·9978062
log 16384 = 4·2144199
log 742 = 2·8704039
log 67502 = 4·8293166
log 67501 = 4·8293102

53. Given $a = 1\frac{1}{2}$, $b = 13\frac{1}{2}$, $C = 65°$; find c, A and B. Given

L. cot 32° 30′ = 10·1958127   L. tan 51° 28′ = 10·0988763
log 2 = ·3010300   L. tan 51° 29′ = 10·0991355

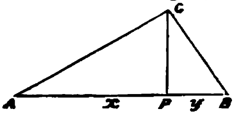
54. In every triangle,

$$\cos^2 A + \cos^2 B + \cos^2 C + 2 \cos A \cos B \, . \cos C - 1 = C.$$

55. In every triangle,

$$\tan \frac{A}{2} = \frac{1}{s - a}\sqrt{\frac{(s - a)(s - b)(s - c)}{s}}$$

56. In the accompanying figure, $C = 90°$, x, y, the segments of AB made by the perpendicular CP; prove that



$$a = \sqrt{y(x + y)}$$
$$b = \sqrt{x(x + y)}$$

57. If the area of a triangle $= \frac{c^2}{4}$, then $C = 90°$.

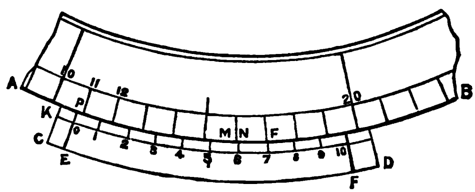
58. Two sides of a triangle are 20 and 12 feet, and the included angle is 120°; find the other angles, having given

log 48 = 1·6812412
L. tan 8° 12′ = 9·1586706
diff. for 60″ = ·0008940.

CHAPTER X.

HEIGHTS AND DISTANCES.

It will here be necessary that the student should have a description of the instruments used in taking altitudes, &c.



1. **The Vernier.**—This is an instrument for subdividing the equal parts that have been made on a circle or straight line.

Thus, if an arc AB be divided into any number of equal parts, and we wish to subdivide these again, then we bring the *vernier* into requisition. Here the *vernier* is the scale CD which slides along AB, revolving round the same centre.

Let the arc AB be divided into any number of equal parts, and suppose it be required to subdivide these again into 10 equal parts. Take EF, measuring 9 of the divisions on the limb AB, and divide it into 10 equal parts. Now if each of the divisions on AB be one inch, then each division of the *vernier* will be $\frac{9}{10}$ths of an inch.

It will be evident that if one of the lines marking a division of the *vernier* coincides with one of those of the limb, the remaining part between the other ends of the divisions will be $\frac{1}{10}$th of an inch.

Or if, as in the figure, the 7th division of the vernier

coincided with a division on the limb, then FM $= 1$ inch. FN $= \frac{9}{10}$ inch $\therefore$ MN $= \frac{1}{10}$ inch, and if A be placed at the zero of the graduated limb, then AP $= 10$ inches $+ \frac{7}{10}$ inches $= 10 \cdot 7$ inches. If the divisions on the limb were degrees, then AP would be $10° \cdot 7$.

Or suppose that the 9th division of the vernier coincided with a division on the graduated limb, and that the zero of the vernier was somewhere between the 11th and 12th divisions of the limb; then AP would be equal to $11 \cdot 9$, or the zero of the vernier would mark $11° \cdot 9$ on the scale.

Generally, if each of the divisions on AB were denoted by x, then each division on the *vernier* would be denoted by $\frac{n-1}{n}x$, where n represents n parts of the limb.

Now let F, the rth division from K, coincide with a division on the *vernier*,

$$\begin{aligned} \text{then KP} &= \text{KF} - \text{FP} \\ &= rx - r\frac{n-1}{n}.x \\ &= rx\left(1 - \frac{n-1}{n}\right) \end{aligned}$$

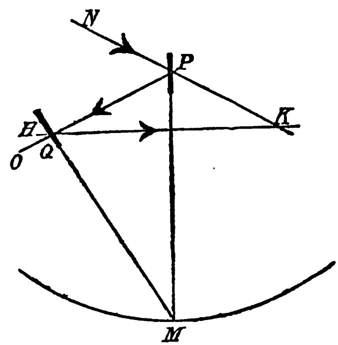
$$\therefore \text{KP} = r.\frac{x}{n}$$

So that if the divisions on the limb were marked *degrees*, then KP $= 7 \times \frac{1}{10}° = \frac{7}{10}° = \cdot 7°$. So that we see KP is the $\frac{7}{10}$ths of a *degree*.

If the *vernier* were to divide each degree into minutes, then the length of it must be $60 - 1$ or $59°$; we then divide these $59°$ into 60 equal parts and proceed as before. The *vernier*, it must be remembered, is equally applicable to the *straight line* as the *circle*.

2. **The Sextant.**—This instrument is for the purpose of measuring the angle between two objects. Navigators find it also very useful in obtaining *latitude* and *longitude*.

It is in the form of a sector of a circle, the arc of which measures, as a rule, 60°, or a sixth part of the whole circumference. It is constructed on the following principle: Let N and H be the two objects, K the eye of the observer. Two mirrors P, Q are inclined to each other at an angle PMQ. A ray of light falling from N on P is reflected on to the mirror Q, and from thence to the eye at K, where it meets the other ray produced.



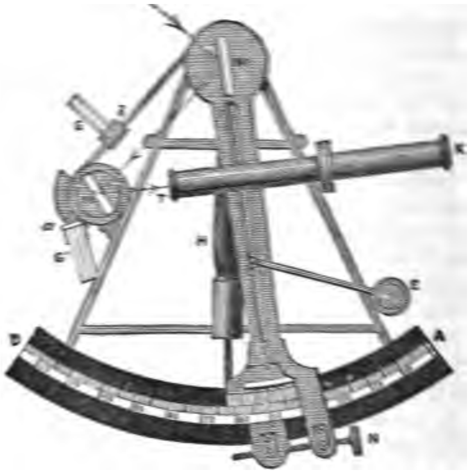
Now the angle PMQ between the mirrors is half the angle NKH required, for by a law of optics, the angle of *incidence* is equal to the angle of *reflection*. Hence the angles QPK and KQO are bisected by the lines PM, QM,

$$\begin{aligned} \text{and } \angle\ \text{NKH} &= \text{PQH} - \text{QPK} \\ &= \text{OQK} - \text{QPK} \\ &= 2\,(\text{OQM} - \text{QPM}) \\ \therefore\ \angle\ \text{NKH} &= 2\ \angle\ \text{PMQ} \end{aligned}$$

Although the sextant contains an arc of 60°, it is graduated up to 120°.

The accompanying figure will give the student an idea of the instrument. The two mirrors are M and M′, TK the telescope, H the handle by which to hold it, II′ the movable index, DA the graduated arc, E an eye-glass to read the vernier, GG′ coloured glasses movable round the axes *b a* respectively. The glasses G are to be placed in the direction of the reflected ray, and thus to prevent the observer's eye from being dazzled by the rays of the sun when taking his altitude. The coloured glasses G′

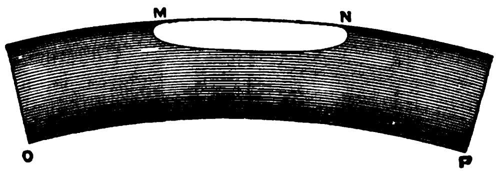
may be placed in the path of the rays proceeding from the other object.



The mirror M′ is silvered at the back of its lower half. When the vernier of the index stands at 0° on the graduated arc, this mirror M′ is placed parallel to the index glass.

If taking the sun's altitude, the observer holds the sextant with the right hand by the handle H, so as to see the horizon through the telescope coincide with the boundary-line separating the silvered from the unsilvered half of the mirror M′. The index is then moved by his other hand until the rays reflected from both glasses enter the telescope at T. The sight being thus taken, he moves the index arm by means of the tangent screw N at its extremity until the boundary-line between the silvered and unsilvered parts of the mirror M′ bisects the sun's figure. The portion of the index arm thus traversed is the measure of the angular distance between the sun

and the horizon. The angle between the mirrors is read off the vernier through the eye-glass E. The angle between the horizon and the sun being double this. we have then found the sun's altitude.

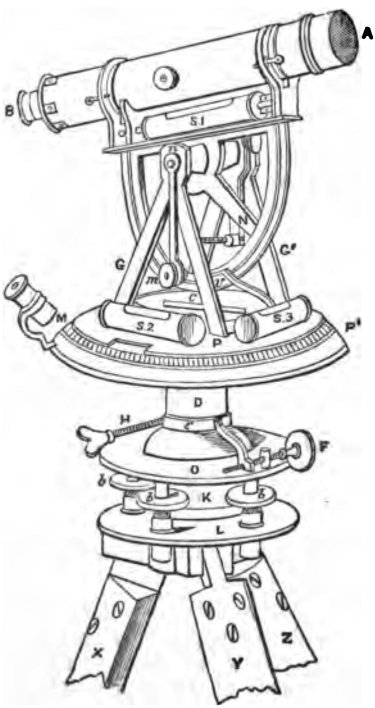


3. **The Spirit Level.**—This is a glass tube of round bore. It is not exactly straight, being in the form of a circular arc of very large radius. The bore is filled with a fluid, generally ether, and the ends are then closed. If placed in a vertical position, with its ends resting on a horizontal plane, the bubble of fluid will remain on the top in the highest possible position, as in the figure. If one end be raised, the bubble MN will gradually approach that end. The spirit level is generally used in a theodolite, and enables the observer to set the instrument in a horizontal position.

4. **The Theodolite.**—The figure on the following page is a representation of the theodolite.

Description.—It consists of two circular plates P P′, called the "limb," of which the upper one, or vernier, turns freely on the lower. They are both connected with the axis D, which consists of two parts—the *external* and *internal*—nicely ground into each other. The latter is secured to the vernier plate P, and the former fixed to the lower or graduated limb P′. The external part is also fitted into a ball at K. The lower limb P′ is usually graduated to half a degree, and is subdivided by the vernier into minutes; these are easily read off by the microscope M, and even quarter-minutes can be discerned by this also. The lower part of the

instrument consists of two parallel plates O and L, which are held together by a ball and socket at K. They are kept parallel by the screws b, b, b.



Underneath these plates is a female screw, adapted to the staff-head, which is connected to the three legs XYZ. These, when shut up, form a round staff, and are rendered portable by rings being put on them, and when opened out form a very firm stand. On the vernier plate are placed two spirit levels s_2, s_3 at right angles to one another; a magnetic compass C is also carried on this plate. The limb is set in a horizontal position by means of the spirit levels.

The frames GG′ support a horizontal axis, which carries a graduated vertical semicircle N, and along the diameter of which is placed a telescope, with a spirit level s, used for placing the telescope in a horizontal position. A vernier v denotes the altitude or depression of an object, and these can be read off by the microscope m fixed as in the figure, revolving round n as a centre. The telescope

can be made to revolve round the axis, carrying this graduated vertical semicircle, without touching the limb P.

In the focus of the eye-glass B of the telescope three lines are placed, formed of a spider's web, one horizontal and the other two crossing it so as to include a very small angle.

Method of Observing with the Theodolite.—Placing the instrument exactly over the station from whence the angles are to be taken, first clamp the horizontal limb firm in any position, and direct the telescope to one of the objects to be observed, keeping it moving until the cross-wires and object coincide; then clamp the upper limb, and by means of its tangent screw make the intersection of the wires bisect the object; now by means of the verniers read off one of them the number of ° ′ ″, and only the ′ ″ of the other, and take the mean of the two readings; next release the upper plate, and direct the telescope to the second object, and clamping it as before, make the cross-wires bisect it; read off the two verniers, as in the last case, and take the mean of the two readings. Subtract the one mean from the other, and the angle thus found will be the angle between the two objects subtended at the station.

Thus, if the angle read off one vernier be

$$\begin{array}{lr}
 & 120^\circ\ 38'\ 20'' \\
\text{and off the other} & \underline{\ldots\ \ 37'\ 50''} \\
 & \underline{2)241^\circ\ 16'\ 10''} \\
\text{mean} = & 120^\circ\ 38'\ \ 5''
\end{array}$$

When the telescope is directed to the other object, if the angle read off the first vernier be

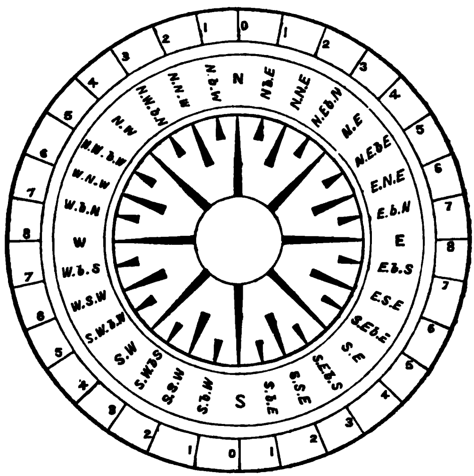
$$\begin{array}{lr}
 & 75^\circ\ 45'\ 30'' \\
\text{and off the other} & \underline{\ldots\ \ 46'\ \ 0''} \\
 & \underline{2)151^\circ\ 31'\ 30''} \\
 & 75^\circ\ 45'\ 45'' \\
\text{then from} & 120^\circ\ 38'\ \ 5'' \\
\text{subtract} & \underline{75^\circ\ 45'\ 45''} \\
 & 44^\circ\ 52'\ 20''
\end{array}$$

and this is the angular distance between the two objects.

The magnetic bearing of an object is found thus:—

Move the upper plate until the magnetic needle points to zero, and at this moment also read from the horizontal plate P. Next turn the upper plate, bisect the object, and read again. The difference between the two readings thus found will be the magnetic bearing of the object.

5. The annexed figure is a representation of a compass card, an inspection of which will make the student acquainted with the meaning of the terms W.b.S., S.S.W., N.E.b.N., &c., which read thus: West by south, south-



south-west, north-east by north, &c. The circle is divided into thirty-two parts, and consequently each quadrant into eight of these equal parts, each of which is called a point and these again subdivided into half and quarter points.

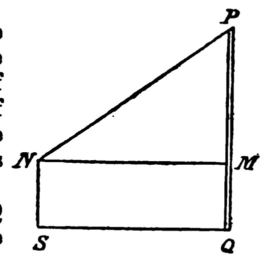
$$\text{A point, therefore,} = \frac{90^\circ}{8} = 11^\circ\ 15'$$

$$\text{A half-point} \qquad = \ 5^\circ\ 37'\ 30''$$

$$\text{A quarter-point} \qquad = \ 2^\circ\ 48'\ 45''$$

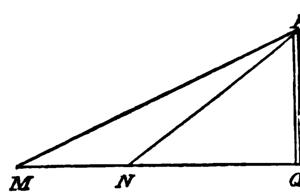
If we speak of a point as 7 points right of north, this is E.b.N.; $7\frac{1}{2}$ points right of north, E.b.N.$\frac{1}{2}$E., or E.$\frac{1}{2}$N.; $6\frac{3}{4}$ points left of south, W.S.W.$\frac{3}{4}$W., or W.b.S.$\frac{1}{4}$S., and so on.

6. *To find the height of an accessible object above a horizontal plane.*



Let PQ be the object whose height is required. A base-line SQ is measured from the foot of the object. NS is the height of the observer's eye above the ground. The angle PNM is then measured, and $\therefore$ PM = MN.tan PNM, add NS or MQ to this, and we get PQ the height of the object.

7. *To find the height and distance of an inaccessible object on a horizontal plane.*

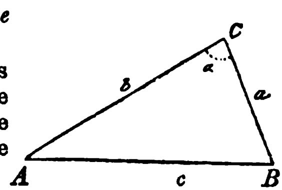


Let PQ be the object whose height and distance from a point M in the same horizontal plane is required. Measure MN directly towards the object, and observe the angles PMQ, PNQ.

Hence, in the triangle PMN, since MN is known and the angles PMN, PNM are also known, we can find PM.

$$\therefore \text{PQ} = \text{PM} \sin \text{PMN}$$
$$\text{and MQ} = \text{PM} \cos \text{PMN}.$$

8. *Two objects are inaccessible to each other on a horizontal plane; to find the distance between them.*



Let A and B be the objects inaccessible to each other. Take a convenient station C, measure the distances AC, CB, and the angle ACB.

This comes under the case "*given two sides and included angle*" in solution of triangles, and we can thus calculate the side AB or c by the formula,

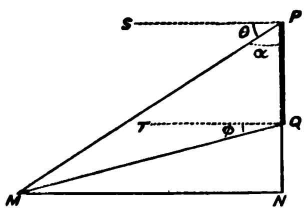
$c = (a+b) \cos \theta$, where θ is an angle,

determined from

$$\sin^2 \theta = \frac{4ab}{(a+b)^2} \cdot \cos^2 \frac{ACB}{2}$$

9. *To find the distance of an object on a horizontal plane from two observations above the plane.*

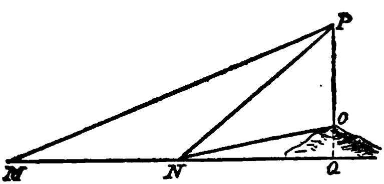
Let M be the object, P and Q two stations above the plane in the same vertical line, at a measured distance from each other. Observe the angular depressions of the object from P and Q—namely, θ and φ; PS and TQ being parallel lines with the horizon. Hence in the triangle MPQ, the side PQ is known; so also are the angles MPQ, MQP, therefore we know MQ.



$$\text{And } \therefore \text{ distance MN} = \text{MQ} \cos \varphi$$
$$\text{,, height QN} = \text{MQ} \sin \varphi.$$

10. *To find the distance of an inaccessible object above a horizontal plane, also the height of the object itself.*

Let two stations MN be taken at a known distance from each other, and in a line with the object. Observe the angles QMP, QNP, the elevations of the top as seen from M and N respectively, also observe the angle QNO, the elevation of the bottom of the object as seen from O.

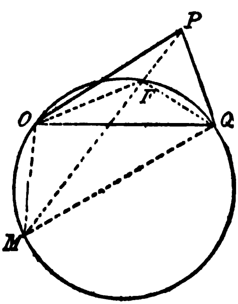


Then in the triangle MNP the side MN is known, and the angles PMN, PNM are also known; hence we can find PN.

$$\begin{aligned} \text{Hence } PQ &= PN \sin \angle PNQ \\ \text{and } \quad NQ &= PN \cos \angle PNQ \\ \text{and } \therefore\ OQ &= NQ \tan \angle ONQ. \end{aligned}$$

Therefore we have found PQ the height of the object above the plane; and since PO = PQ – OQ, we have found PO the height of the object itself.

11. *If three points in a horizontal plane be at given distances, it is required to find their distances from another point in the same plane.*



Let OPQ be the three objects whose distances OP, PQ, QO are known. Let M be the other object. It is required to find the distances MO, MP, MQ. About the triangle MOQ describe a circle, cutting MP in F. Join OF, FQ.

From M, observe the angles OMP, PMQ. Hence the angles OQF, FOQ are known, being respectively equal to these.

In the triangle OFQ, OQ is known and the two angles adjacent to this side, and therefore we can calculate the sides OF, FQ.

The sides OP, PQ, QO being proportional to the sines of the opposite angles, these can therefore be calculated, and hence we know the angles POF and PQF.

In the triangle POF, we have therefore two sides PO, OF, and included angle known; therefore we can calculate the angle OPF. Similarly we determine the angle QPF.

The three angles of the triangles POM, PQM are known, and we can therefore calculate the distances MO. MP, MQ, from Case 2, solution of oblique-angled triangles.

EXERCISES.

1. A man $5\frac{1}{2}$ feet high stands 125 feet from the foot of an object and observes its elevation to be 52° 34′. Find its height.

2. The elevation of an object 900 feet from its base is found to be $28\frac{1}{2}$°, whereas its elevation at the bottom is 52°. Find its height.

3. I wish to find the distance between two objects, and take a station 4,080 yards from one and 5,368 yards from the other, and find that they subtend an angle of 54° 32′ 40″ at this point. Find their distance from each other.

4. In order to find the d istance between two inaccessible objects, I take up two stations 1,500 feet from each other, and observe that each object makes with one extremity of the line angles of 118° 20′, and 46° 14′, while at the other end they make angles of 88° 48 and 33° 12′. Find their distance apart.

5. I take up a station inside three objects equally distant from one another, and observe that a line joining two of them would subtend an angle of 140° 20′, and a line joining two others would subtend an angle of 88° 30′ at the point of observation. Suppose that each object was 180 feet distant from the others; find my distance from each.

6. To find the height of an inaccessible object above a horizontal plane, I take three stations whose distances from each other in the same straight line are x, y, z. At each station I observe the angular elevation to be α, β, γ, respectively. Show that

$$p = \left\{ \frac{xyz}{x \cot^2 \alpha - z \cot^2 \beta + y \cot^2 \gamma} \right\}^{\frac{1}{2}}$$

where p is the height of the object above the plane.

7. The sides of a triangle are 2, 3, and 4. Find tne cosine of the obtuse angle without logarithms. (Exam., 1876.)

8. From Plymouth the Lizard is distant 54·44 miles; from the Lizard, Start Point is distant 17·15 miles; and from Start Point to Plymouth is 23·31 miles.

From Eddystone Light	Plymouth	bears	N. 25° 4′ 5″ E.
	Lizard	„	S. 70° 13′ 15″ W.
	Start	„	N. 83° 52′ 20″ E.

Find the distance of Eddystone from each of the others.

9. Neglecting refraction, and assuming the diameter of the earth to be 41,800,000 feet, what is the greatest distance that a peak 19,000 feet high could be visible across the sea from aonther peak 9.000 feet high? (Exam., 1875.)

10. Explain the principle of the vernier. If a scale marks tenths

of an inch, how must the vernier be constructed to enable us to read accurately to the $\frac{1}{160}$th part of an inch?

11. Give a short description of the theodolite, showing how to use it so as to escape errors arising from the vertical circle not being truly centred.

12. The altitude of an object is observed with a reflecting sextant and an artificial horizon. Supposing the sextant to be held very near the artificial horizon, would that introduce any geometrical error into the observation? Would it do so if the object were a star? Illustrate your proof by a diagram.

13. Show that in the reflecting sextant the observed angle is double the angle which the index moves through. How would you take an altitude with a sextant and an artificial horizon? The limb being graduated to minutes, and the vernier reading to 10ths, give a sketch showing the actual reading on the limb and vernier, supposing the altitude to be 37° 16′ 36″.

14. If in Question 6 the three stations A, B, C are directly under the object, and the angular elevation at B be double that at A, and that at C three times that at A, show that

$$p = \frac{x\sqrt{(x+y)(3y-x)}}{2y}$$

15. A ship sights another A° from the north sailing in a parallel direction to its own. In p hours its bearing was B°, and in q hours afterwards C° from the north. In what direction were the ships sailing? If the course of the vessels make an angle ϕ with the north, show then that

$$\frac{\sin(\phi - C)}{\sin(\phi - A)} = \frac{p}{q} \cdot \frac{\sin(B - A)}{\sin(C - B)}$$

16. A tower 66⅔ yards high, standing on the bank of a river, has a flagstaff 10 yards high on its summit which subtends an angle of the same number of degrees with an observer on the opposite bank as it does with an object 2 yards high at its base. Find the breadth of the river.

17. Two towns are north and south of each other, and distant 2½ miles; a person in a balloon observes their angles of depression to be 45° and 60°: after the balloon has proceeded horizontally in a S.E. direction for 6 miles, the angles of depression are 22½° and 30°. Find the height of the balloon.

18. At sea a vessel sights a lighthouse bearing N.N.E., and after sailing E.b.S. for 7 miles, the lighthouse bears N.W.b.N. Calculate the distance of the lighthouse from the ship at each point of observation.

19. From the top of a pillar 40 feet high I find that another pillar 60 yards high subtends an angle of 36°. What is the horizontal distance of the last pillar?

20. A privateer lies 8 miles from the mouth of a harbour bearing E.N.E., observes a ship leaving the harbour and bearing S.E.b.E. and sailing at the rate of 7 knots per hour. In what direction must the privateer sail in order to overtake the ship in $2\frac{1}{2}$ hours; at what rate will she sail, and what distance will have to be run?

21. At sea I observe two lighthouses, A and B, and when in line with them I observe B bearing S.E.$\frac{1}{2}$E.; when I have sailed $10\frac{1}{2}$ miles in the same direction I observe that B bears S.W.b.S. How does the other object A bear, and what distance off is it, supposing that the distance between A and B be 25 miles?

22. In a balloon 2,200 yards above the surface of the earth, I observe the angle of depression of an object below, measured from the sea horizon, is 10° 30′. How many miles from the object will the balloon descend, supposing it to descend in a vertical direction?

23. The summit of an object 2 miles high is just visible from another mountain summit 3 miles high; how far apart are they?

24. The altitude of a balloon is observed to be 22° 15′, and going forward $133\frac{1}{3}$ yards, its altitude is observed to be 48° 20′. Find the height of the balloon.

25. Three stations O, P, Q are in a straight line, OP = 240 feet, PQ = 75 yards. The angles of elevation of an inaccessible object M at each station are respectively 72° 19′, 78° 16′, and 70° 10′. Find the height of the object M.

26. From A, B, two stations 2,088·63 yards apart, I observe two towers, C, D, and make the following measurements:—

$$\angle\, BAC = 139^\circ\ 15'\ 45'' \qquad \angle\, DAB = 53^\circ\ 30'\ 23''$$
$$\angle\, ABC = 31^\circ\ 49'\ 0'' \qquad \angle\, ABD = 114^\circ\ 24'\ 55''$$

From these data find the distance between the two towers.

27. Standing at a certain station, I can just see the summit of a mountain 1,284 yards high. Ascending in a balloon, I observe the angular depression of the summit of the mountain to be $2\frac{1}{4}$°. Find the height of the balloon at the time of observation, supposing the earth's diameter to be 7,986·4 miles.

28. Two forts are distant from a lighthouse 1·0352 and ·96101 miles, and they subtend an angle of 59° 59′ 22″ at the lighthouse. Find their distance apart.

29. The height of a balloon is 1,682·67 feet, and its angular elevations from two stations are found to be 15° 17′ 18″ and 62° 30′ 20″. Find the distance between the two objects.

30. A tower 6 points right of south from an observer throws a shadow in a north-easterly direction. Suppose the angular elevation of the tower to be 45°, and its height 61·23 feet; find the length of the shadow.

31. From the top of a church steeple I observe the angular depressions of two objects in the same line with the steeple to be θ and ϕ. If the height of the steeple be denoted by the expression, $x \dfrac{\tan\theta \cdot \tan\phi}{\tan\theta - \tan\phi}$, find the distance between the two objects.

32. At a station due south of an object, I find its elevation to be 30°, and at another station westward of it the angle of elevation is 18°; if the distance of the object from the second station be 5 miles, find the height of the object.

33. The angular height of a tower was observed to be 15°, and 100 yards nearer it was observed to be 30°. Find its actual height. (Exam., 1876.)

34. ACB is the diameter of a circle, and CD a radius at right angles to it. Bisect CD in E, join AE, and produce it to meet the circle in F; let fall FG perpendicular to CD. Show that the triangle FGC has its sides as 3 : 4 : 5. (Exam., 1876.)

35. Three objects, A, B, C, form an isosceles triangle whose vertex is B, and whose angles are as the numbers 4, 1, 1: a person walking from A to C measures a base AD $= a$ feet, and observes the angle BDC; he then advances to E b feet farther, and finds the angle BEC $= 180° -$ BDC. Find the sides of the triangle.

36. Describe the vernier, and give a sketch of a scale and vernier showing the reading 1·7, the unit representing one of the divisions of the scale. (Exam., 1876.)

37. A person wishing to ascertain the horizontal distance of two inaccessible objects from each other, can find no point from which they are visible together; he finds, however, two stations, the distance between which he can determine, from which the objects may be separately seen. Explain what observations and measurements it will be necessary for him to make in order for him to effect his purpose.

38. The top of a tower is visible from A, B, C in the same horizontal line; at each station the angular distance of the top of the tower from each of the other two stations is observed. Given the distance between A and B, and the height of the tower; find the distance of C from each of the other stations, and from the tower.

CHAPTER XI.

TRIANGLES AND REGULAR POLYGONS IN AND ABOUT A CIRCLE, &c.

1. From Art. 6, Chapter VIII., we know that the area of a triangle $= \frac{1}{2}\, bc, \sin A \quad \ldots\ldots 1$

$$\text{or} = \sqrt{s\,(s-a)\,(s-b)\,(s-c)} \cdot \ldots\ldots 2$$

$$\text{now } b = c \,.\, \frac{\sin B}{\sin C}$$

Substituting this in (1) and denoting the area of a triangle by Δ,

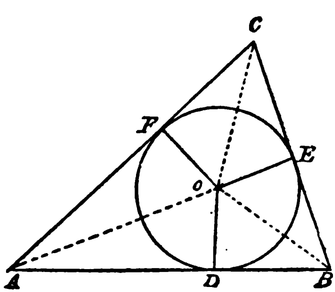
$$\text{we have } \Delta = \frac{c^2}{2} \cdot \frac{\sin A . \sin B}{\sin C}$$

$$\text{or } \Delta = \frac{b^2}{2} \cdot \frac{\sin A . \sin C}{\sin B}$$

$$\text{or } \Delta = \frac{a^2}{2} \cdot \frac{\sin B . \sin C}{\sin A}$$

Either of these expressions gives the area of a triangle when one side and two angles are known.

2. *To find an expression for the radius of the inscribed and circumscribed circles of a triangle.*



In the accompanying figure, where DEF is the circle inscribed in the triangle ABC, O its centre, join OF, OE, &c. ; denote each of the lines OF, OE, OD by r, and the sides of the triangle a, b, c, opposite their corresponding angles.

Now area of triangle AOB $= \frac{cr}{2}$

" BOC $= \frac{ar}{2}$

" AOC $= \frac{br}{2}$

$\therefore$ area of whole triangle ABC $= r \cdot \frac{a+b+c}{2}$

or $\Delta = r \cdot s$

$\therefore r = \frac{\Delta}{s}$

or radius of inscribed circle $= \frac{\text{area of triangle}}{\text{semi-perimeter}}$

3. Again, since AB = AD + DB

$$\text{and AD} = r \cot \frac{A}{2}$$

$$\text{DB} = r \cot \frac{B}{2}$$

$$\therefore \text{AB or } c = r\left(\cot \frac{A}{2} + \cot \frac{B}{2}\right)$$

$$= r\left\{\frac{\cos \frac{A}{2}}{\sin \frac{A}{2}} + \frac{\cos \frac{B}{2}}{\sin \frac{B}{2}}\right\}$$

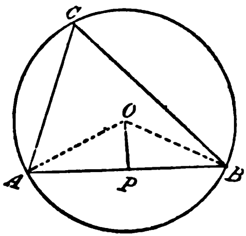
$$= r\,\frac{\sin \frac{A+B}{2}}{\sin \frac{A}{2} . \sin \frac{B}{2}}$$

$$\therefore r = c . \frac{\sin \frac{A}{2} . \sin \frac{B}{2}}{\sin \frac{A+B}{2}}$$

$$\text{Similarly } r = b.\frac{\sin\frac{A}{2}.\sin\frac{C}{2}}{\sin\frac{A+C}{2}}$$

$$\text{,, } \quad r = a.\frac{\sin\frac{B}{2}.\sin\frac{C}{2}}{\sin\frac{B+C}{2}}$$

It should be noticed by the student that the lines OA, OB, OC bisect the angles of the triangle (Euclid IV., 4).



4. Let a circle be circumscribed about the triangle ABC. From centre O let fall OP perpendicular to AB, join OA, OB. AB is bisected in P (Euclid III., 3), and the angle AOB is also bisected. Hence $\angle$ AOP = $\angle$ C (Euclid III. 20). Let AO and OB be denoted by R.

$$\text{Now AP or } \frac{c}{2} = \text{R}.\sin \text{C}$$

$$\therefore \text{R} = \frac{c}{2 \sin \text{C}} \quad \ldots\ldots\ldots (1)$$

$$\text{Similarly R} = \frac{b}{2 \sin \text{B}} \quad \ldots\ldots\ldots (2)$$

$$\text{R} = \frac{a}{2 \sin \text{A}} \quad \ldots\ldots\ldots (3)$$

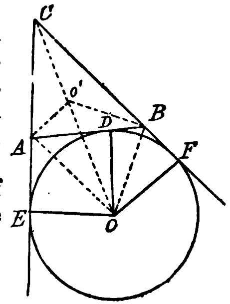
but from Art. 5, Chap. VIII., we know that

$$\sin \text{A} = \frac{2\,\Delta}{bc}$$

Hence, from Equation 3,

$$\text{R} = \frac{a}{\frac{4\,\Delta}{bc}} = \frac{abc}{4\,\Delta}$$

5. *To find an expression for the radius of the escribed circle to a triangle.*



Let DEF be the escribed circle to the triangle ABC, O its centre; complete the figure. Let r_a r_b r_c be the radii of the escribed circles which touch the sides a, b, c, respectively.

Area of figure CAOB = area of triangle OAC + area of triangle OBC

$$= r_c.\frac{b}{2} + r_c.\frac{a}{2}$$

also area of figure CAOB

= area of triangle ABC + area of triangle ABO

$$= \Delta + r_c.\frac{c}{2}.$$

$$\therefore r_c.\frac{b}{2} + r_c\frac{a}{2} = \Delta + r_c.\frac{c}{2}$$

$$\therefore r_c\frac{(b+a-c)}{2} = \Delta$$

$$r_c(s-c) = \Delta \therefore r_c = \frac{\Delta}{s-c}$$

$$\text{Similarly } r_b = \frac{\Delta}{s-b}$$

$$r_a = \frac{\Delta}{s-a}$$

Another expression for r_a, r_b, r_c can also be got by the following method.

If the angles CAB and CBA were bisected by lines AO′, BO′, the lines AO, AO′ would be at right angles to one another; so also would BO and BO′.

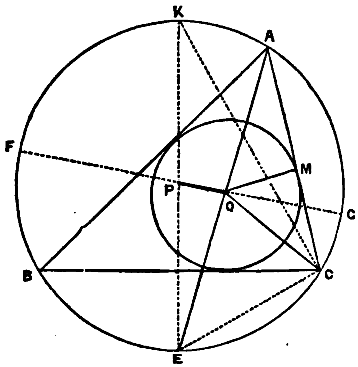
$$\begin{aligned}\text{now } AB &= AD + DB \\ &= r_c \cot \angle DAO + r_c \cot \angle DBO \\ &= r_c \tan \frac{A}{2} + r_c \tan \frac{B}{2} \\ &= r_c \left(\tan \frac{A}{2} + \tan \frac{B}{2}\right)\end{aligned}$$

$$\therefore\ r_c = c . \frac{\cos \frac{A}{2} . \cos \frac{B}{2}}{\sin \frac{A+B}{2}}$$

$$\text{Similarly } r_b = b . \frac{\cos \frac{A}{2} . \cos \frac{C}{2}}{\sin \frac{A+C}{2}}$$

$$r_a = a . \frac{\cos \frac{B}{2} . \cos \frac{C}{2}}{\sin \frac{B+C}{2}}$$

6. *To find an expression for the distance between the centres of the inscribed and circumscribed circles.*



Let P and Q be the centres of circumscribed and inscribed circles respectively; join PQ, AQ, CQ; produce AQ to E; join EP, and produce it to K; join CK, CE, and produce PQ to meet the circle in F and G; draw QM per. to AC.

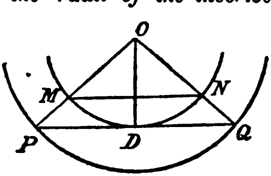
Now since the angles A and C of the triangle are bisected,

$$\therefore\ \angle\ EQC = \frac{A}{2} + \frac{C}{2}$$

$$\text{also } \angle \text{ECQ} = \frac{C}{2} + \angle \text{BCE}$$
$$= \frac{C}{2} + \frac{A}{2}.$$
$$\therefore \angle \text{EQC} = \angle \text{ECQ}$$
$$\therefore \text{side EQ} = \text{EC}$$
$$\text{Now EQ} = \text{EC} = \text{EK} \sin \text{EKC}$$
$$= 2\,\text{R} \sin \frac{A}{2}$$
$$\therefore 2\,\text{R} = \frac{\text{EQ}}{\sin \frac{A}{2}}$$
$$\text{Again, QM} = \text{QA} \sin \frac{A}{2}$$
$$\therefore r = \text{QA} \sin \frac{A}{2}$$
$$\therefore 2\,\text{R}r = \frac{\text{EQ}}{\sin \frac{A}{2}}\,\text{QA}.\sin \frac{A}{2}$$
$$= \text{EQ}.\text{QA}$$
$$\text{Now EQ}.\text{QA} = \text{FQ}.\text{QG}$$
$$= (\text{R}+\text{PQ})(\text{R}-\text{PQ})$$
$$\therefore 2\,\text{R}r = \text{R}^2 - \text{PQ}^2$$
$$\therefore \text{PQ} = \sqrt{\text{R}^2 - 2\text{R}r}$$

The student can in a similar manner find an expression for the distance between the centres of the inscribed and escribed circles. Suppose this distance to be denoted by d, then $d = \sqrt{\text{R}^2 + 2\text{R}r_a}$

7. *To find expressions for the radii of the inscribed and circumscribed circles of a regular polygon.*



Let PQ be the side of the inscribed regular polygon of the outer circle, and the side of the circumscribed regular polygon of the inner circle.

Let OD $= r$, and OP $=$ R, and PQ $= a$.

Now the angle POQ is the nth part of 360° $\therefore$ the angle POD $= \frac{\pi}{n}$.

$$PD = PO \sin \angle POD$$

$$\therefore \frac{a}{2} = R.\sin\frac{\pi}{n}$$

$$\therefore R = \frac{a}{2\sin\frac{\pi}{n}}$$

$$\text{Also } PD = OD.\tan\frac{\pi}{n}$$

$$\frac{a}{2} = r\tan\frac{\pi}{n}$$

$$\therefore r = \frac{a}{2\tan\frac{\pi}{n}}$$

8. *To find an expression for the area of the circumscribed regular polygon of n sides.*

In the last figure the area of the triangle OPQ

$$= PD.DO$$

$$= \frac{a}{2}.r$$

$$= r\tan\frac{\pi}{n}.r$$

$$= r^2\tan\frac{\pi}{n}$$

but the area of the polygon is n times the area of this triangle.

$$\therefore \text{area of circumscribed polygon} = nr^2\tan\frac{\pi}{n}$$

9. *To find an expression for the area of the inscribed regular polygon (circle same radius r).*

$$\triangle \text{ POQ} = \tfrac{1}{2} \text{ PO}.\text{OQ} \sin \angle \text{ POQ}$$
$$= \frac{r^2}{2} \sin \frac{2\pi}{n}$$
$$\therefore \triangle \text{ inscribed polygon} = \frac{nr^2}{2}.\sin \frac{2\pi}{n}$$

Or, if we call R the radius of circumscribed circle, and r the radius of inscribed circle to the polygon,

$$\text{then } \triangle \text{ polygon} = \frac{n\text{R}^2}{2} \sin \frac{2\pi}{n}$$
$$\text{and } \triangle \text{ polygon} = nr^2 \tan \frac{\pi}{n}$$

10. *To find the area of a polygon of n sides in terms of a side.*

In last figure

$$\triangle \text{ polygon} = n.\text{PD}.\text{DO}$$
$$= n \frac{a}{2}.r$$
$$= n \frac{a}{2} . \frac{a}{2} \cot \frac{\pi}{n} = \frac{na^2}{4} \cot \frac{\pi}{n}$$

11. *To find an expression for the area of a circle.*

Now, if the number of sides of the polygon be indefinitely increased, the perimeter of the polygon will ultimately coincide with the circumference of the circle, and if we called the sides of the polygon a, b, c, &c.,

$$\text{then } \triangle \text{ polygon} = \frac{a}{2}.r + \frac{b}{2}.r + \frac{c}{2}.r + \ldots\ldots$$
$$= \frac{a+b+c+\ldots\ldots}{2}.r$$

but when the polygon coincides with the circle, $a + b + c + \ldots\ldots = 2\pi r$;

$$\therefore \triangle \text{ circle} = \frac{2\pi r}{2}.r$$
$$\text{or } \triangle \text{ circle} = \pi r^2$$

12. We can get this by another method,

$$\text{since } \triangle \text{ polygon} = nr^2 \tan\frac{\pi}{n}$$

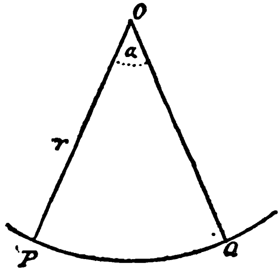
$$= \frac{\pi r^2}{\cos\frac{\pi}{n}} \cdot \frac{\sin\frac{\pi}{n}}{\frac{\pi}{n}}$$

and if, as before, n increased indefinitely, the area of the polygon continually approaches the area of the circle, and we know from Chapter VII., Art. 26, that when n is increased indefinitely

$$\cos\frac{\pi}{n} = 1, \text{ and } \frac{\sin\frac{\pi}{n}}{\frac{\pi}{n}} = 1$$

Hence $\triangle$ circle $= \pi r^2$

13. *To find an expression for the area of a sector of a circle.*



Let OPQ be a sector of a circle with radius r, and let a be the circular measure of $\angle$POQ;

then $\triangle$ sector : $\triangle$ circle : : $a : 2\pi$

$$\therefore \triangle \text{ sector} = \triangle \text{ circle} \times \frac{a}{2\pi}$$

$$= \pi r^2 \times \frac{a}{2\pi}$$

$$= \frac{r^2 a}{2}$$

or since a : ang. unit :: PQ : r

$$\therefore \text{PQ} = ra$$

$$\text{Hence } \triangle \text{ sector} = \frac{\text{PQ}}{2} . r$$

$$= \text{half arc} \times \text{radius.}$$

14. *To compare the areas of the inscribed and circumscribed regular polygons.*

Let Δ = area of inscribed polygon

Δ' = „ circumscribed „

$$\therefore \frac{\Delta}{\Delta'} = \frac{\frac{nr^2}{2}\sin\frac{2\pi}{n}}{nr^2\tan\frac{\pi}{n}}$$

$$= \frac{\frac{nr^2}{2}.2\sin\frac{\pi}{n}.\cos\frac{\pi}{n}}{nr^2.\frac{\sin\frac{\pi}{n}}{\cos\frac{\pi}{n}}}$$

$$\therefore \frac{\Delta}{\Delta'} = \cos^2\frac{\pi}{n}$$

Before we close this chapter we will work a few examples in full, in order to give the student a clearer notion of the method in which such problems are solved.

1. Prove the following:—

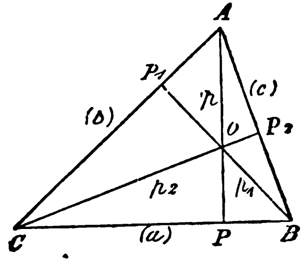
$$\frac{2}{dD} = \frac{1}{bc} + \frac{1}{ac} + \frac{1}{ab}$$

where d = diameter of inscribed circle

and D = „ circumscribed „

$$\text{Now } \frac{2}{dD} = \frac{2}{2r\ 2R} = \frac{1}{r\ 2R} = \frac{1}{\frac{\Delta}{s}\ \frac{2abc}{4\Delta}}$$

$$= \frac{2s}{abc} = \frac{a+b+c}{abc} = \frac{1}{bc} + \frac{1}{ac} + \frac{1}{ab}$$



2. If $p\ p_1\ p_2$ be the perpendiculars let fall from the angles of a triangle on the opposite sides,

then $\dfrac{p^2}{p_1\,p_2} = \dfrac{bc}{a^2}$

The triangles APB, CP_2B are equiangular, hence

$$\frac{c}{a} = \frac{p}{p_2}$$

similarly $\dfrac{b}{a} = \dfrac{p}{p_1}$ $\therefore$ $\dfrac{bc}{a^2} = \dfrac{p^2}{p_1 p_2}$

3. In the last figure prove

$$\frac{OB.OC}{BA.BC} + \frac{OB.OA}{BC.CA} + \frac{AO.OC}{AB.BC} = 1$$

Let $\angle BOC = \alpha$, $\angle BOA = \beta$, and $\angle AOC = \gamma$

Let Δ = area of triangle ABC

Now $\dfrac{OB.OC}{2} \sin \alpha$ = area triangle OBC

$$\therefore\ OB.OC = \frac{2 \text{ area } OBC}{\sin \alpha}$$

$$\text{also } AB.AC = \frac{2 \text{ area } ABC}{\sin A} = \frac{2\Delta}{\sin \alpha}$$

for $\sin \alpha = \sin A$, since the quadrilateral AP_1OP_2 is inscribable in a circle, and therefore $\sin \alpha = \sin \angle P_1OP_2 = \sin A$.

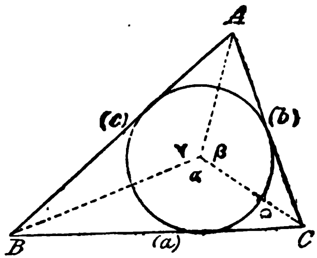
$$\text{Hence } \frac{OB.OC}{AB.AC} = \frac{\text{area } OBC}{\Delta}$$

$$\text{similarly } \frac{OB.OA}{BC.CA} = \frac{\text{area } AOB}{\Delta}$$

$$\text{and } \frac{AO.OC}{AB.BC} = \frac{\text{area } AOC}{\Delta}$$

$$\therefore \frac{OB.OC}{AB.AC} + \frac{OB.OA}{BC.CA} + \frac{AO.OC}{AB.BC} =$$

$$\frac{\text{area OBC} + \text{area AOB} + \text{area AOC}}{\Delta} = \frac{\Delta}{\Delta} = 1$$



4. A circle is inscribed in a triangle and α, β, γ are the angles at the centre opposite the sides a, b, c, respectively; prove then that

$$4 \sin \alpha, \sin \beta, \sin \gamma = \sin A + \sin B + \sin C.$$

Since the angles A, B. C are bisected,

$$\therefore 4 \sin \alpha, \sin \beta, \sin \gamma,$$

$$= 4 \sin \frac{B+C}{2} . \sin \frac{A+C}{2} . \sin \frac{A+B}{2}$$

$$= 4 \sin \frac{B+C}{2} . \tfrac{1}{2} \left\{ \cos \frac{B-C}{2} - \cos \left(A + \frac{B+C}{2} \right) \right\}$$

$$= 2 \sin \frac{B+C}{2} . \cos \frac{B-C}{2} - 2 \sin \frac{B+C}{2} . \cos \left(A + \frac{B+C}{2} \right)$$

$$= \sin B + \sin C - (\sin (A+B+C) - \sin A)$$

$$= \sin A + \sin B + \sin C, \text{ since } \sin 180° = 0$$

5. In a plane triangle, if d and D be the diameters of the inscribed and circumscribed circles, prove that

$$d + D = a \cot A + b \cot B + c \cot C$$

Now $a \cot A + b \cot B + c \cot C =$

$$a\frac{\cos A}{\sin A} + b\frac{\cos B}{\sin B} + c\frac{\cos C}{\sin C} =$$

$$\frac{a}{\sin A}(\cos A + \cos B + \cos C) \text{ since } \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$= \frac{a}{\sin A}\left\{2\cos\frac{A+B}{2}\cos\frac{A-B}{2} - \cos(A+B)\right\}$$

$$= \frac{a}{\sin A}\left\{2\cos\frac{A+B}{2}.\cos\frac{A-B}{2} - 2\cos^2\frac{A+B}{2} + 1\right\}$$

$$= \frac{a}{\sin A}\left\{2\cos\frac{A+B}{2}\left(\cos\frac{A-B}{2} - \cos\frac{A+B}{2}\right) + 1\right\}$$

$$= \frac{a}{\sin A}\left\{4\sin\frac{C}{2}\sin\frac{A}{2}\sin\frac{B}{2} + 1\right\}$$

$$= 2a\frac{\left(1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)}{4\sin\frac{A}{2}\cos\frac{A}{2}}$$

$$= 2a\left\{\frac{1}{2\sin A} + \frac{\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}}\right\}$$

$$= 2\left\{\frac{a}{2\sin A} + a\frac{\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}}\right\} = 2(R + r)$$

6. The radii of the inscribed and circumscribed circles of a plane triangle being r and R, prove

$$2Rr = \frac{abc}{a+b+c} \quad \text{and}$$

$$\frac{r}{R} = 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

By Art. 3 of this chapter we know that

$$r = a \frac{\sin \frac{B}{2} \sin \frac{C}{2}}{\cos \frac{A}{2}}$$

and also that

$$R = \frac{a}{2 \sin A}$$

Hence, by multiplication,

$$2\,Rr = a^2 \frac{\sin \frac{B}{2} \sin \frac{C}{2}}{2 \sin \frac{A}{2} \cos^2 \frac{A}{2}}$$

$$= \frac{a^2}{2} \sqrt{\frac{\frac{(s-a)(s-c)}{ac} \cdot \frac{(s-a)(s-b)}{ab}}{\frac{(s-b)(s-c)}{bc}}} \times \frac{bc}{s(s-a)}$$

$$= \frac{a^2}{2} \cdot \frac{(s-a)}{a} \times \frac{bc}{s(s-a)}$$

$$= \frac{abc}{2s} = \frac{abc}{a+b+c}$$

$$\text{also } \frac{r}{R} = \frac{a \sin \frac{B}{2} \sin \frac{C}{2}}{\cos \frac{A}{2}} \times \frac{4 \sin \frac{A}{2} \cos \frac{A}{2}}{a}$$

$$\therefore \frac{r}{R} = 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$$

7. If r be the radius of the inscribed circle of a triangle, and r_a, r_b, r_c be the radii of the escribed circles opposite the angles A, B, C respectively, prove that

$$rr_ar_br_c = \text{area}^2 \text{ of triangle.}$$

$$\text{Now } r = \frac{\Delta}{s}$$

$$r_a = \frac{\Delta}{s - a}$$

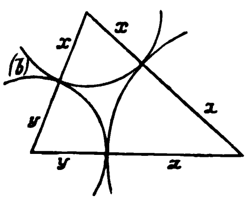
$$r_b = \frac{\Delta}{s - b}$$

$$r_c = \frac{\Delta}{s - c}$$

$$\therefore\ r r_a r_b r_c = \frac{\Delta^4}{s(s-a)(s-b)(s-c)} = \frac{\Delta^4}{\Delta^2} = \Delta^2$$

8. If x, y, and z be the radii of three circles which touch externally, prove that the area of the triangle formed by joining their centres

$$= \sqrt{x\,y\,z\,(x + y + z)}$$



In the accompanying figure it can be seen that since

$$\Delta = \sqrt{s\,(s-a)\,(s-b)\,(s-c)}$$

$$\text{and } s = x + y + z$$

$$s - a = y$$

$$s - b = z$$

$$s - c = x$$

$$\therefore\ \Delta = \sqrt{x\,y\,z\,(x + y + z)}$$

Exercises.

1. If three circles touch externally, having radii x, y, z respectively, prove that the tangents drawn from the points of contact meet in a point whose distance from any of the points of contact is

$$\sqrt{\frac{xyz}{x + y + z}}.$$

2. If Δ Δ' Δ'', &c., denote the areas of several triangles described about the same circle, prove that

$$\Delta : \Delta' : \Delta'' :: p : p' : p''$$

where $p, p' p''$ denote the sum of the sides of each triangle respectively.

3. If r be the radius of a circle inscribed in a triangle, and r_a, r_b, r_c be the radii of the circles inscribed between this circle and the sides containing the angles A, B, C respectively, prove that

$$r = \sqrt{r_a r_b} + \sqrt{r_b r_c} + \sqrt{r_a r_c}$$

4. Prove that the sum of the perpendiculars let fall from the angles of a triangle on the opposite sides $= 2(R + r)$, where R and r are the respective radii of the circumscribed and inscribed circles.

5. Prove that $R = \dfrac{a \cos A + b \cos B + c \cos C}{4 \sin A, \sin B, \sin C}$, where R is the radius of the circumscribed circle to a plane triangle.

6. In the figure to Art. 7 prove that $D + d = x \cot \frac{\pi}{2n}$, where D, d are the diameters of the circumscribed and inscribed circles, x the length of one side of the polygon of n sides.

7. Prove that the area of a triangle is equal to either of the following expressions :—

$$\frac{b^2 - c^2}{2} \frac{\sin B . \sin C}{\sin (B - C)}$$

$$\text{or} \quad \frac{2\,abc}{a + b + c} . \cos \frac{A}{2} . \cos \frac{B}{2} . \cos \frac{C}{2}$$

8. In the figure to Art. 6, if AQ meet BC in M′, prove that

$$\frac{AQ}{QM} = \frac{\cos (B - C)}{\cos A}$$

9. Prove that $\tan \frac{A}{2} = \sqrt{\frac{rr_a}{r_b r_c}}$, where r, r_a, r_b, r_c are the radii of the inscribed and escribed circles of a triangle.

10. If the centres of the three escribed circles of a triangle be joined, and a, b, c denote the sides of the original triangle, with r the radius of its inscribed circle, prove that the area of this triangle so formed is $\frac{abc}{2r}$

11. Let s be a segment of a circle with radius unity, c its chord, and v its versed sine, and let $\frac{2v}{c} = z$; prove that

$$\frac{s}{2} = z(1 + 1) - z^3\left(3 + \frac{1}{3}\right) + z^5\left(5 + \frac{1}{5}\right) - z^7\left(7 + \frac{1}{7}\right) +$$

12. If $\tan 2A = \dfrac{pr + qs}{pq - rs}$

and $\tan 2B = \dfrac{pr - qs}{pq + rs}$, prove that

$\tan (A + B) = \dfrac{r}{q}$ or $-\dfrac{q}{r}$

$\tan (A - B) = \dfrac{s}{p}$ or $-\dfrac{p}{s}$

13. Given the chord c, and versed sine v of a circular arc; show how to find the angle which it subtends from the centre, and also the area of the segment.

14. CD subtends an angle α at A, and also at B; and CA is at right angles to BD. Prove that AB = CD cot α.

15. Show that the perimeter of a regular polygon of n sides inscribed in a circle is less than the perimeter of a regular polygon of $n + 1$ sides inscribed in the same circle.

16. In a right-angled triangle, if one side be double the other, then $r_a + r_c = 2r_b$, where r_a, r_b, r_c are the radii of the escribed circles.

17. Prove that $\tan \dfrac{A}{2} = \dfrac{r}{s - a}$

18. In a right-angled triangle, where C is the right angle and r the radius of the inscribed circle, prove that

$$b = \frac{2r(a - r)}{a - 2r}$$

$$\text{also } b = r\sqrt{2}\,\frac{\sin\left(45^\circ + \dfrac{A}{2}\right)}{\sin\dfrac{A}{2}}$$

19. Prove that the area of a circle is denoted by the expression, $\dfrac{c^2}{4\pi}$, where c = circumference, and $\pi = 3 \cdot 14159$.

20. Prove that the area of a sector of a circle is equal to $\dfrac{A^\circ}{360^\circ}\pi r^2$, where A is the angle at the centre of the circle, of radius r, subtended by the arc of the sector.

21. If r, r' be the radii of two concentric circles, then show that the area of the *annulus* or portion enclosed between the circumferences is equal to $(r + r')(r - r')\pi$.

22. If Δ denote the area of a segment of a circle, show that

$$\Delta = \frac{r}{2}\left(a - r\sin\frac{2\pi}{n}\right)$$

$$\text{or } \Delta = \frac{r}{2}\left(a + r\sin\frac{2\pi}{n}\right)$$

where a is the arc, r the radius, according as the segment is less or greater than a semicircle, n = number of sides of inscribed regular polygon.

23. A circle has an equilateral triangle inscribed in it; a circle is inscribed in the triangle which also has an equilateral triangle inscribed in it, and so on. Find the sums of the perimeters and the areas of all the circles and triangles.

24. The area of a regular polygon of n sides in a circle : the area of another regular polygon of $3n$ sides in the same circle : : $p : q$. Find the values of the angles subtended by a side of each at the centre.

25. Find the side of a regular quindecagon in a circle of radius r.

26. Find an expression for the side of a regular hexagon inscribed in a circle; also one for the side of a regular dodecagon in the same circle, and compare their perimeters and areas.

27. Compare the sides and areas of the squares and regular octagons described in and about a circle.

28. If r be the radius of a circle inscribed in a triangle; $r_1\ r_2\ r_3$ the radii of three other circles touching the sides and sides produced of the same triangle; prove then that

$$\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$$

29. In a right-angled triangle a perpendicular is drawn from the right angle to the opposite side; show that the areas of the circles inscribed in the triangles made by it are proportional to the corresponding segments of the side.

30. If lines be drawn from all the angles of a polygon to any point, then the products of the sines of the angles so formed, taken alternately, are equal.

31. A circle is inscribed in an equilateral triangle, an equilateral triangle in the circle, a circle in this last triangle, and so on; prove that the radius of any one circle is equal to the sum of the radii of all those within it.

32. If an equilateral polygon of 2^n sides be inscribed in a circle whose radius is 1, prove that

$$\text{one side} = \sqrt{2 - \sqrt{2 + \sqrt{2 + \&c.}}}$$

the radical sign being repeated n times.

33. Prove that the area of a regular polygon inscribed in a circle

is a mean proportional between the areas of an inscribed and circumscribed regular polygon of half the number of sides.

34. If a point be assumed in a regular polygon of n sides, and perpendiculars let fall from it to each of the sides or sides produced, then if P be the sum of these perpendiculars, and r the radius of the inscribed circle, $P : r :: n : 1$.

35. The square of the side of a regular pentagon inscribed in a circle is equal to the sum of the squares of the sides of a regular hexagon and decagon in the same circle.

36. If x be the side of an equilateral triangle inscribed in a circle, and y the side of an inscribed square,

$$\text{then } x : y :: \sqrt{3} : \sqrt{2}$$

$$\text{and } \triangle \text{ triangle} : \triangle \text{ square} :: 3\sqrt{3} : 8$$

37. If P and p be the perpendiculars from the extremities of the base of a triangle, upon the line bisecting the vertical angle at distances D and d from it, then prove that

$$\begin{aligned} Pp &= (s - a)(s - b) \\ Dd &= s(s - c) \\ \text{and area} &= Pd = pD \end{aligned}$$

38. The diameter of the circumscribing circle

$$= \frac{2s}{\sin A + \sin B + \sin C} = \sqrt[3]{\frac{abc}{\sin A \cdot \sin B \cdot \sin C}}$$

39. The diameter of the inscribed circle

$$= 2s \cdot \tan\frac{A}{2} \cdot \tan\frac{B}{2} \cdot \tan\frac{C}{2}$$

40. If $\triangle \cdot \triangle'$ be the areas of the two circles, then

$$\frac{\triangle}{\triangle'} = \frac{r^2}{r'^2} = \frac{D^2}{D'^2} = \frac{C^2}{C'^2} \text{ where C is the circumference.}$$

41. The area of a regular polygon circumscribed about a circle is an harmonical mean between the areas of an inscribed regular polygon of the same number of sides, and of a circumscribed regular polygon of half the number of sides.

42. Let r be the radius of inscribed circle in a triangle, and r_1, r_2, r_3 the radii of the circles touching severally one of the sides a, b, c externally, and the other two internally; show that $rr_1 + r_2r_3 = bc$.

43. The area of a regular polygon inscribed in a circle : the area of a similar figure circumscribed about it :: 3 : 4. Find the number of sides.

44. Find the area included between two regular polygons of the same number of sides, one being inscribed in, and the other circumscribed about, a circle of given radius r.

ANSWERS.

CHAPTER I.

1. $42^g\ 34'\ 56''$.
2. $72^g\ 50'\ 61{\cdot}727''$.
   $33^g\ 50'\ 70{\cdot}987''$.
   $53^g\ 89'\ 83{\cdot}79''$.
3. 60°, 9°, $81^\circ\ 40'\ 38{\cdot}343''$.
4. $n = 11$.
5. 8 and 12.
6. $39^g\ 28'\ 53{\cdot}7''$.
7. 8 and 12.
8. $\frac{(n-1)}{n}.200^g$, $\left(\frac{n-1}{n}\right)180^\circ$
9. 20°, 40°.
10. 36°, 18°.
11. $\frac{9}{10}.\frac{y}{x}$
13. $(n-2)\ 200^g$, $(n-2)\ 180^\circ$.
14. $63\frac{3}{19}^\circ$, $56\frac{16}{19}^\circ$, 60°.
16. 18°, 30°, 36°, 45°, $85^\circ\ 30'$, $22^\circ\ 54'\ 14{\cdot}436''$.
17. (*a*) $35^g\ 29'\ 53{\cdot}7\dot{0}\dot{3}''$. (*b*) $12^g\ 50'$. (*c*) 25^g. (*d*) $83\frac{1}{3}^g$ (*e*) $87^g\ 50'$.
18. 9.
19. 5 and 10.

CHAPTER II.

1. $63{\cdot}661977238^g$.
3. $2\pi r$, $3{\cdot}14159$.
4. $\frac{\pi}{6}$, $\frac{2\pi}{15}$, $\frac{5\pi}{2}$, $\frac{\pi}{180}$, $\frac{10\pi}{180\times 60\times 60}$, $\frac{\pi^2}{200}$.
5. 45°, 270°, $\frac{180}{\pi}$, $\frac{11}{21}$, $\frac{180(\pi+1)}{\pi}$, $\frac{135}{\pi}$.
6. $\pi - \frac{19\pi^2}{1800}$
7. $\frac{5\pi}{19}$.
8. $\frac{1}{32}$ of 360°.
9. $\frac{x}{2\pi}$.
10. $\frac{10}{\pi}$.

12. $\frac{2\pi}{3}$.

13. $A = \frac{1}{n} \cdot \frac{a}{r}$.

14. $\frac{17\frac{1882}{3600} \cdot \pi}{180}$.

15. $\frac{315}{2\pi}$ inches.

17. $\frac{\pi}{10}$.

18. 95,050,000 miles nearly.

CHAPTER III.

2. $\cos A = \sqrt{\frac{2}{3}}$

$\tan A = \frac{1}{\sqrt{2}}$

$\cot A = \sqrt{2}$

$\sin A = \frac{1}{\sqrt{3}}$

$\operatorname{cosec} A = \sqrt{3}$.

3. $\sin A = \frac{3}{5}$

$\cos A = \frac{4}{5}$

$\tan A = \frac{3}{4}$

$\cot A = \frac{4}{3}$

$\sec A = \frac{5}{4}$.

4. $\sin A = \frac{12}{13}$

$\tan A = \frac{12}{5}$

$\cot A = \frac{5}{12}$

$\sec A = \frac{13}{5}$

$\operatorname{cosec} A = \frac{13}{12}$.

5. $\sin A = \frac{1}{\sqrt{x^2+1}}$

11. ·69.

12. $\sin A = \frac{1}{\sqrt{5}}$

$\cos A = \frac{2}{\sqrt{5}}$

16. $\sin A = \frac{\sqrt{7}}{4}$

$\sec A = \frac{\sqrt{26}}{5}$

$\cos A = \frac{5}{\sqrt{34}}$

$\cos A = \frac{29}{65}$.

19. $\sin A = \frac{-3 \pm \sqrt{10}}{2}$

20. $\tan A = \frac{2xy}{x^2 - y^2}$

CHAPTER IV.

1. Unity, 2,
   $\sin a = 1$ when $a = 90°$.
2. $10° \, 2' \, 15''$, $100° \, 2' \, 15''$.
7. See Chap. VI., Art. 8.
8. See Art. 2.
9. $\tan 15° = 2 - \sqrt{3}$.
   $\tan 75° = \cot 15°$.
   $\tan 105° = -\cot 15°$.
   $\tan 165° = -\tan 15°$.
   $\tan 195° = \tan 15°$.
   $\tan 255° = \cot 15°$.
   $\tan 285° = -\cot 15°$.
   $\tan 345° = -\tan 15°$.
   $\tan -1000° = \tan 80°$.

12. $\sin 210^\circ = -\sin 30^\circ$.
$\sin -120^\circ = -\sin 60^\circ$.
$\sin 165^\circ = \sin 15^\circ$.

13. In 1st quad, as A increases, $\sin A - \cos A$ increases, and is positive when $A > 45^\circ$, and negative when $A < 45^\circ$. Similarly for 2nd quad, &c.

15. $0^\circ, 90^\circ, 180^\circ, 270^\circ$.

19. $2n\pi \pm \frac{2\pi}{3}$.

20. $n\pi \pm \frac{\pi}{6}$.

21. $\frac{2n\pi}{3} \pm \frac{2\pi}{9}$.

23. $a = 45^\circ$

24. $\pm \cos \beta$.

25. $-\cot a$.

27. $\tan \theta = \frac{2n+1}{4} \pm \frac{\sqrt{4n^2+4n-15}}{4}$
n being any integer.

28. $\sec \left\{ (6n+3)\frac{\pi}{2} + a \right\} = \frac{(-1)^n}{a}$

29. $\cos A = \frac{1}{2}$, $\sin A = \frac{\sqrt{3}}{2}$, &c.

30. $\sin A = \pm \frac{m}{\sqrt{1+m^2}}$

31. $\sec B = \pm \sqrt{\frac{m^2-n^2}{1-n^2}}$

CHAPTER V.

18. $\sin 8a = o$
$\therefore\ 8a = n\pi$
$\therefore\ a = \frac{n\pi}{8}$.

27. $\pm x = \frac{\sqrt{(b-c)^2+a^2}-b+c}{\sqrt{(b+c)^2+a^2}-b-c}$.

28. $\sin 2\theta = \pm(\frac{3}{4} - 4m^2 + 2m)$, m being any integer.

29. $\cos 2a = \frac{1 \pm \sqrt{3}}{2}$.

36. $\tan(a - 2x) = \frac{n-m}{n+m} \cdot \tan a$.

38. $\cos(\theta + \phi) = \frac{b^2-a^2}{b^2+a^2}$
$\cos(\theta - \phi) = \frac{a^2+b^2-2}{2}$.

40. $y^2(3p+x) = (2p+x)^2(p-x)$.

49. $\frac{1}{m} + \frac{1}{n} = \frac{1}{a} + \frac{1}{b}$.

53. (a). $\cos a = \frac{bc \pm a\sqrt{a^2+b^2-c^2}}{a^2+b^2}$
(b) $\cos(a-\beta) = \frac{m^2+n^2-2}{2}$.

54. (a). $a = 2n\pi \pm \frac{2\pi}{3}$
or $a = \frac{n\pi}{2}$
(b) $a + \frac{\pi}{4} = n\pi \pm \frac{\pi}{3}$.

59. $a = 36^\circ$ or 108°.

60. $x = \sin 4a$.

CHAPTER VI.

1. $\sin 22\frac{1}{2}^\circ = \frac{\sqrt{2-\sqrt{2}}}{2}$

   $\cos 22\frac{1}{2}^\circ = \frac{\sqrt{2+\sqrt{2}}}{2}$

   $\tan 22\frac{1}{2}^\circ = \sqrt{2} - 1.$

2. $\tan \ 7\frac{1}{2}^\circ = \frac{\sqrt{2}-1}{\sqrt{2}+\sqrt{3}}.$

3. $\tan 142\frac{1}{2}^\circ = 2 + \sqrt{2}$
   $- \sqrt{3}\ (1+\sqrt{2}).$

4. $a = \frac{(2n+1)}{4}\pi$

   or $a = 2n\pi \pm \frac{2\pi}{3}$

5. $2n\pi \pm a.$

6. $\sin 9^\circ = \frac{\sqrt{3+\sqrt{5}} - \sqrt{5-\sqrt{5}}}{4}$

   $\cos 9^\circ = \frac{\sqrt{3+\sqrt{5}} + \sqrt{5-\sqrt{5}}}{4}$

7. $\sin 3^\circ = \sin\ (18^\circ - 15^\circ).$

9. $\cos\ a = \frac{\sqrt{3}}{2}.$

10. 270° and 450°.

11. 270° and 450°.

13. $\sin 165^\circ + \cos 165^\circ$

    $= -\sqrt{1+\sin 330^\circ}$

    $\sin 165 - \cos 165^\circ$

    $= \sqrt{1-\sin 330^\circ}$

    $\therefore \sin 165^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}}$

    $\cos 165^\circ = -\frac{\sqrt{3}+1}{2\sqrt{2}}.$

14. Four values

    $\sin\frac{a}{2} + \cos\frac{a}{2} = +\sqrt{1+\sin A}$

    $\sin\frac{a}{2} - \cos\frac{a}{2} = +\sqrt{1-\sin A.}$

16. 450° and 630°.

17. $\cos 11^\circ\ 15'$

    $= \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}$

CHAPTER VII.

1. $\frac{1}{3}\{8\log 2 + 4\log 3 - 7\}.$

2. $-4, (\sqrt{3})^{\frac{34}{3}}, -2, -1.$

3. 3, 2.

4. See Arts. 1 and 5.

6. $N = a^{P}.$

7. $\frac{1}{8}.$

8. 343.

9. 3.

10. $-1\cdot5906306; -2\cdot2544674.$

11. $\frac{1}{m}\left\{\begin{array}{l} n\log a + q\log(b+x) \\ \quad -p\log(a-x) \\ \quad -p\log(a+x) \end{array}\right\}$

14. If $\log_{\frac{1}{1000}} 2 = x \therefore 2 = \left(\frac{1}{1000}\right)^x$

    $\therefore x(\log 1 - \log 1000) = \log 2.$

16. (*a*) $7\cdot2725$ yrs. (*b*) $43\cdot46$

17. $\log 2 - \log 3$; $\log 3 - \log 2$; $\log 2 + 1 - \log 3$; $\log 3 - 1 - \log 2$; &c.

18. $\bar{4}\cdot4771213$.

19. $0\cdot8573326$.

20. $\cdot0616835428$.

21. $\frac{1}{4}(2\log 2 - 3\log 7), -\cdot322205$.

22. $2\cdot06818$, $2\cdot19312$.

23. $1\cdot532660$.

24. (a). $2\theta - a = 2n\pi \pm \frac{\pi}{3}$

(b). $\cos\frac{5\theta}{2} = 0$, or $\cos\theta = 0$

or $\sin\frac{\theta}{2} = 0$

$\therefore \frac{5\theta}{2} = 90°$, or $\theta = 90°$ or $\frac{\theta}{2} = 180°$.

25. (a). $m\cos(a - \phi) = x$.

or $-\cos(a + \phi) = x$.

(b) $(x+1)a = 2n\pi \pm \left(\frac{\pi}{2} - \frac{a}{2}\right)$

26. (a). $x = \dfrac{4\log b}{2\log c + \log b - 3\log a}$

(b). $x = 2$.

(c). $x = 4\frac{1}{2}$, $y = 1\frac{1}{2}$.

28. L. $\sin 1' = 6\cdot4637261$.

30. $\cdot5172818$.

31. 1, 2.

32. 20 digits.

35. $\cdot5875153$.

36. $9\cdot5050352$.

37. $9\cdot9269586$, $9\cdot8009511$.

38. $A = 30°\ 21'\ 30\cdot3''$.

39. $\cot\theta = \frac{1}{\tan\theta} = \tan\theta^{-1}$

$\therefore -1 = \log_{\tan\theta}\cot\theta$.

40. $10\cdot1792358$.

CHAPTER VIII.

6. $\sin a = \dfrac{a^2 - 1}{a^2 + 1}$.

7. The triangle is here right-angled $\therefore$ C $= 90°$

and $\therefore \tan\theta = \frac{b}{a}$

$\therefore \tan\frac{\theta}{2} = -\dfrac{a \pm \sqrt{a^2 + b^2}}{b}$

9. $\tan^2\frac{a}{2} = \dfrac{1+m}{1-m}\cdot\tan^2\frac{\beta}{2}$

11. 2 area $\left(\dfrac{ab + bc + ac}{abc}\right)$

12. $\dfrac{a + b + c}{\cot\frac{A}{2} + \cot\frac{B}{2}}$

24. $\frac{a}{b} = \frac{\sin A}{\sin B}$.

30. $\cos\theta = \sqrt{1 + \cos a}$.

CHAPTER IX.

2. $24°\ 20'\ 26\cdot2''$.

3. $A = 28°\ 4'\ 21\cdot96''$.

5. Where $\cos\phi = \frac{c}{b}$

6. $\frac{b^2}{2}\sin 2A$.

7. $A = 117°\ 38'\ 45''$, $B = 27°\ 38'\ 45''$

8. $74\cdot397$ square feet.

10. Area $= 59750\cdot41$ square feet; if the segments be x, y, then $\frac{x}{y} = \frac{\tan C}{\tan A}$.

11. AP=401·4377 feet.

14. From the triangle BAQ, the side AQ can be found, and from △ ABP the side PA can be found; hence, in the triangle PQA, since two sides and included angle are given, PQ can be got.

16. $x = (a + b) \tan 15°$, where x = height of staff.

20. 216 square feet.

21. A=83° 24′ 48″, B=36° 35′ 12″.

22. Distance=2,088 feet. Height=508 feet.

23. 248 yards.

26. B=27° 59′ 43″, C=44° 57′ 4″, c=46·357.

29. c = 33·249, b = 31·76, A= 17° 12′ 51″.

30. 50° 33′ 45″, 461·5504, 597·6171.

31. 92° 9′ 23″, 42° 49′ 42″, 287·035

32. 51° 15′ 35″, 37° 21′ 25″, 305·296.

39. Area=3,727 acres, 3 roods, 21 perches nearly.

40. $\cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$

42. tan A=±1, tan B=±2, tan C=±3.

49. $\tan B = \frac{p \sin A}{\sqrt{q^2 - p^2 \sin^2 A}}$

50. $L. \tan \frac{A}{2} = 9{\cdot}6733937$.

52. c=67·502, A = 71° 33′ 15″, B=60° 2′ 45″.

53. A=108° 58′ 6″, B=6° 1′ 54″.

58. A=38° 12′ 47·6″ B=21° 47′ 12·4″.

CHAPTER X.

1. 56·27 yards.
2. 282·87 yards.
3. 2 miles 958 yards.
4. 761·9 yards.
5. 42·11, 43·86, 21·28 yards.
7. $-\frac{1}{4}$
8. 13·1, 44·2, 27·2 miles.
9. 1431043 feet.
16. 107·23 feet.
17. 3 miles.
18. 5·953, 8·257 miles.
19. 89·54 yards.
20. Rate 9 knots, dist=22·9 miles; must sail E.b.S.½S.
21. Bearing 64° 29′ W. of N.; distance=27½ miles.
22. About 6 miles.
23. 280 miles.
24. 257·3 feet.
25. 286·321 yards.
26. 33336·4 feet.
27. 3 miles.
28. 1 mile.
29. 1 mile.

30. 80 feet.

31. x.

32. $\dfrac{5}{\sqrt{2+2\sqrt{5}}}$ miles.

33. 50 yards.

35. Side $=\dfrac{2a+b}{\sqrt{3}}$

CHAPTER XI.

3. Sum of perimeters of circles to infinity $=4\pi r$, where r is the radius of the outer circle.

Sum of perimeters of triangles where "a" is a side of the outer one $=6a$.

Areas of circles $=\dfrac{4\pi r^2}{3}$

Areas of triangles $=\dfrac{a^2}{\sqrt{3}}$

24. $\sin\dfrac{2\pi}{n}=\dfrac{3\sqrt{3(q-p)}}{2}\{1-(q-p)\}$

$\sin\dfrac{2\pi}{3n}=\dfrac{\sqrt{3(q-p)}}{2}$.

25. If x be a side, then

$$x^2=2r^2\left(1-\cos\frac{2\pi}{15}\right).$$

26. Side of hexagon $=r$

Side of dodecagon $=\dfrac{r}{2}\cdot\dfrac{1}{\cos 15^\circ}$

27. Side of inscribed square $=r\sqrt{2}$

Side of circumscribed square $=2r$

Side² of inscribed octagon $=r^2\sqrt{2}(\sqrt{2}-1)$

Side of circumscribed octagon $2r(\sqrt{2}-1)$

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